Robust Maximum Lifetime Routing and Energy Allocation in Wireless Sensor Networks *

Ioannis Ch. Paschalidis † Ruomin Wu ‡

May 14, 2012

Abstract

We consider the maximum lifetime routing problem in wireless sensor networks in two settings: (a) when nodes' initial energy is given, and (b) when it is subject to optimization. The optimal solution and objective value provide optimal flows and the corresponding predicted lifetime, respectively. We stipulate that there is uncertainty in various network parameters (available energy and energy depletion rates). In setting (a) we show that for specific, yet typical network topologies, the actual network lifetime will reach the predicted value with a probability that converges to zero as the number of nodes grows large. In setting (b) the same result holds for all topologies. We develop a series of robust problem formulations, ranging from pessimistic to optimistic. A set of parameters enable the tuning of the conservatism of the formulation to obtain network flows with a desirably high probability that the corresponding lifetime prediction is achieved. We establish a number of properties for the robust network flows and energy allocations and provide numerical results to highlight the trade-off between predicted lifetime and the probability it is achieved. Further, we analyze an interesting limiting regime of massively deployed sensor networks and essentially solve a continuous version of the problem.

Keywords: Wireless sensor networks, routing, maximum lifetime, robust optimization, linear programming.

1 Introduction

Wireless Sensor Networks (WSNETs) have emerged as an exciting new paradigm of inexpensive, easily deployable, completely untethered device networks that

*Research partially supported by the NSF under grant EFRI-0735974, by the DOE under grant DE-FG52-06NA27490, by the ARO under grant W911NF-09-1-0492, and by the ODDR&E MURI10 program under grant N00014-10-1-0952.

†Department of Electrical and Computer Engineering, the Division of Systems Engineering, and the Center for Information and Systems Engineering, Boston University, 8 St. Mary's St., Boston, MA 02215, e-mail: yannisp@bu.edu, url: http://ionia.bu.edu/.

‡Center for Information and Systems Engineering, Boston University, e-mail: romie88@gmail.com.

enable the automated and intelligent monitoring and control of physical systems. WSNET nodes can be equipped with a variety of sensors, have a built-in radio to communicate with each other, are powered by batteries, and have limited information storage and processing capabilities. WSNETs can be useful in a plethora of applications including industrial and building automation, health monitoring, wildlife monitoring, and asset and personnel tracking [RLP06]. Battery technology, however, remains a critical bottleneck. In many applications one would like to use the WSNET for long periods, often years without changing batteries. As a result, energy conservation is a primary concern and aggressive optimization becomes indispensable.

In this paper we focus on the problem of selecting an optimal strategy for routing packets from data-collecting sensor nodes to a set of gateways (or sinks) in order to minimize the rate at which energy is consumed, or equivalently, to maximize the lifetime of the network. We consider two situations: (*i*) when the initial energy of every node is given and (*ii*) when it is also subject to optimization given an overall energy budget. Routing, of course, has received quite a bit of attention in WSNETs. Various aspects of the problem have been considered in [BE81, ME98, GCNB99, RM99, LH01, WLBW01, LHB+01, TCS01, ALR03, LP08], which mostly focus on finding a single path from origin to destination. A more static view is adopted in [CT04], followed by [GK05], and [NM05], which provide a linear programming formulation for optimizing average flows between nodes.

Our starting point is the flow optimizing formulation of [CT04] and [NM05]. A different but equivalent formulation using optimal control ideas is in [WC05]. Key data to solve this problem include the total available energy at the nodes and the energy consumption rates. These quantities are hardly known with any degree of certainty or accuracy. Yet, they affect both the optimal flows and the corresponding optimal objective value, that is, the *predicted network lifetime*. The latter value will in fact be equal to the actual network lifetime if all problem data are known with certainty. We note that both these quantities are quite important for the network designer. The predicted network lifetime is useful for planning purposes and the optimal flows indicate how routing should be done to achieve such a lifetime.

Uncertainty, though, renders the predicted lifetime overly optimistic. For the case without energy allocation, we show that for specific, yet typical, topologies including linear and two-dimensional grid-like networks, the actual lifetime will reach the predicted value with a probability that converges to zero as the number of nodes grows large. This suggests that the predicted network lifetime is not a particularly useful estimate under uncertainty.

For the energy allocation case, we show the same result without any topological assumptions. We also find that uncertainty impacts the optimal policy as well, and one needs to use a different set of "robust" flows to protect against uncertainty. To that end, we develop a series of alternative *robust problem formulations*, ranging from pessimistic to optimistic. A set of parameters enable the tuning of the conservatism of the formulation with a desirably high probability that the corresponding lifetime prediction will be achieved—a *life-*

time guarantee probability. Our robust formulations are based on recent work in robust linear programming in [BS04] and [PK05]. However, the problem we consider has special structure which we exploit to establish a number of interesting properties. Robust optimization has in general received a lot of attention lately and has found applications in many areas. It started with [Soy73] with more recent contributions in [BTN99] and [BS04].

To gain more insight, we consider maximum lifetime routing with energy allocation in a continuous setting of *massively dense* WSNETs. Related limiting regimes have previously been considered in [LHB$^+$01, Jac04, TT06]. For a single point source and a single point sink we show that the optimal route is a straight line from the source to the sink. For multiple sources and sinks, we show that sources send their flows to the closest sink, again over a straight line.

The rest of the paper is organized as follows. In Sec. 2 we tackle the maximum lifetime routing problem without energy allocation, introducing robust formulations and characterizing their solutions. Sec. 3 incorporates the energy allocation into the problem. In Sec. 4, we develop the continuous version of the problem with energy allocation. Numerical examples are in Sec. 5. Conclusions are in Sec. 6.

2 Maximum lifetime routing without node energy allocation

We represent a WSNET as a directed graph $\mathscr{G}(\mathscr{N},\mathscr{A})$ where $\mathscr{N}$ is the node set and $\mathscr{A}$ the set of directed links (i,j) with $i,j \in \mathscr{N}$. Link (i,j) exists if and only if $j \in \mathscr{S}_i$, where $\mathscr{S}_i$ is the set of nodes that can be reached by i. Each node i has an initial battery energy of E_i and consumes e^t_{ij} per data unit to transmit to j while j consumes e^r_{ij} per data unit to receive from i. We assume that the nodes are able to relay packets and to adjust the transmit power level to the minimum required in order to reach the intended receiver. *Origin nodes* (or *sources*) $\mathscr{O}$ include all $i \in \mathscr{N}$ with a positive (constant) information generation rate Q_i. $\mathscr{D}$ is the set of *sink nodes* (or *sinks*) responsible for collecting all data. Assume $\mathscr{O} \cap \mathscr{D} = \emptyset$; we refer to nodes in $\mathscr{N} \setminus \mathscr{D}$ simply as *sensor nodes*.

Every source node seeks to send its data to one of the sinks, not necessarily the same one for each data unit generated. To that end, node i may use multiple other nodes as relays. Let q_{ij} be the information transmission rate from i to j. We write $\mathbf{q}$ for the vector of all q_{ij}'s. [1] Note that routing and power control are intrinsically coupled since the power level is adjusted depending on the choice of the next hop.

In the sequel, we only consider the energy spent for communications since this is the dominant energy consumption term in WSNETs (see [SHC$^+$04]). Additional energy consumption terms could be incorporated into e^t_{ij}, e^r_{ij}. For example, a sensing/processing energy cost at transmissions or receptions per

[1] We use bold letters to denote vectors and all vectors are assumed to be column vectors unless explicitly stated otherwise.

data unit can be incorporated into e^t_{ij} and e^r_{ij}. We also assume that e^t_{ij} is monotonically increasing with the distance between i and j. Finally, sink nodes are assumed to be powered by line power.

The *lifetime* of a sensor node i under a given set of flows $\mathbf{q}$ is given by

$$T_i(\mathbf{q}) = \frac{E_i}{\sum_{j\in\mathscr{S}_i} e^t_{ij} q_{ij} + \sum_{j|i\in\mathscr{S}_j} e^r_{ji} q_{ji}}, \ \forall i \in \mathscr{N} \setminus \mathscr{D}.$$

Define the *network lifetime* under flow $\mathbf{q}$ as the minimum lifetime over all nodes, i.e.,

$$T_{net}(\mathbf{q}) \triangleq \min_{i\in\mathscr{N}\setminus\mathscr{D}} T_i(\mathbf{q}).$$

The network lifetime is equivalent to the earliest time a sensor node runs out of energy.

2.1 Problem formulations

The maximum lifetime routing problem without node energy allocation is the problem of selecting flows $\mathbf{q}$ to maximize $T_{net}(\mathbf{q})$. Letting $\hat{q}_{ij} = q_{ij}T$ denote the amount of information transmitted from i to j over the lifetime T, [CT04] formulated the problem as a linear program:

$$\begin{aligned}
\max \quad & T && (1)\\
s.t. \quad & \sum_{j|i\in\mathscr{S}_j} \hat{q}_{ji} + Q_i T = \sum_{j\in\mathscr{S}_i} \hat{q}_{ij}, \ \forall i \in \mathscr{N} \setminus \mathscr{D}, && (2)\\
& \sum_{j\in\mathscr{S}_i} e^t_{ij}\hat{q}_{ij} + \sum_{j|i\in\mathscr{S}_j} e^r_{ji}\hat{q}_{ji} \le E_i, \ \forall i \in \mathscr{N} \setminus \mathscr{D}, && (3)\\
& T \ge 0, \ \hat{q}_{ij} \ge 0, \ \forall i \in \mathscr{N}, \ \forall j \in \mathscr{S}_i, && (4)
\end{aligned}$$

where the decision variables are T and the $\hat{q}_{ij}$'s. [2] The first set of constraints correspond to flow conservation and the second set of constraints follows from the definition of lifetime. We note that this formulation can also account for the energy consumed while the node's radio is listening. Specifically, we can add $e_i^{\text{ON}}\lambda_i T$ to the lefthand side of (3), where e_i^{ON} is the energy consumption rate by the radio while listening and λ_i is the fraction of time node i is "awake" and listening. We refer to (1) as the *nominal* problem. Note that it is always feasible if for every sensor node there exists a path to a sink node. We assume that this will always be the case. We note that problem (1) can be solved in a distributed manner using subgradient optimization techniques for the dual [ML06]. This is appealing for WSNET applications. Here, however, we concentrate on the impact of uncertainty and do not focus on distributed solution approaches. It can be also argued that in several application contexts a distributed approach

[2] On a notational remark, we will use $\hat{\mathbf{q}}$ to denote flow over the lifetime T and $\mathbf{q}$ to denote flow per unit of time. Thus, when we refer to an optimal solution $\mathbf{q}^*$ (resp. $\hat{\mathbf{q}}^*$) of (1) we mean optimal flow per unit of time (resp. over the lifetime).

is not critical since (1) is solved during a planning/deployment stage of the WSNET.

The data for the nominal problem are e^t_{ij}, e^r_{ij}, and E_i and these affect both the optimal solution and the optimal value. As these may be uncertain, we model them as symmetrically bounded nonnegative random variables (r.v.'s) with ranges given by: $e^t_{ij} \in [\bar{e}^t_{ij} - \Delta e^t_{ij}, \bar{e}^t_{ij} + \Delta e^t_{ij}]$, $e^r_{ij} \in [\bar{e}^r_{ij} - \Delta e^r_{ij}, \bar{e}^r_{ij} + \Delta e^r_{ij}]$, and $E_i \in [\bar{E}_i - \Delta E_i, \bar{E}_i + \Delta E_i]$. We will call $\bar{e}^t_{ij}$, $\bar{e}^r_{ij}$, and $\bar{E}_i$ the nominal values and assume that they are the means of the corresponding r.v.'s. The values Δe^t_{ij}, Δe^r_{ij}, and ΔE_i represent the maximum deviations from the mean which are assumed to be identical left and right from the mean (hence, the term *symmetrically* bounded r.v's). These deviations are defined so that all r.v.'s have positive support. We also define the *uncertainty sets* $J^t_i \triangleq \{j | \Delta e^t_{ij} > 0, j \in \mathscr{S}_i\}$ and $J^r_i \triangleq \{j | \Delta e^r_{ji} > 0, i \in \mathscr{S}_j\}$, $\forall i \in \mathscr{N} \setminus \mathscr{D}$.

Due to data uncertainty, the optimal solution of (1) may not be feasible. It can be easily seen that the following *worst-case* formulation guarantees feasibility for any realization of the data:

$$\max \quad T \tag{5}$$

$$\begin{aligned} s.t. \quad & (2),\ (4), \\ & \sum_{j \in \mathscr{S}_i} \bar{e}^t_{ij} \hat{q}_{ij} + \sum_{j \in J^t_i} \Delta e^t_{ij} \hat{q}_{ij} + \sum_{j | i \in \mathscr{S}_j} \bar{e}^r_{ji} \hat{q}_{ji} \\ & + \sum_{j \in J^r_i} \Delta e^r_{ji} \hat{q}_{ji} \le \bar{E}_i - \Delta E_i,\ \forall i \in \mathscr{N} \setminus \mathscr{D}. \end{aligned} \tag{6}$$

We refer to the above as the *fat* problem. By construction, its optimal solution is feasible for any data realization but it may be overly conservative. Intuitively, the probability that all parameters take their "extreme" value should be small, thus, motivating a less conservative formulation.

We introduce the *uncertainty budget* $\Gamma^e_i \in [0, |J^t_i| + |J^r_i|]$ for every sensor node i and define the *restricted uncertainty set* $\mathscr{R}_i(\Gamma^e_i)$ as

$$\begin{aligned} \mathscr{R}_i(\Gamma^e_i) = \{ e^t_{ij}, e^r_{ji} \mid e^t_{ij} \in [\bar{e}^t_{ij} - \Delta e^t_{ij}, \bar{e}^t_{ij} + \Delta e^t_{ij}], e^r_{ij} \in [\bar{e}^r_{ij} - \Delta e^r_{ij}, \bar{e}^r_{ij} + \Delta e^r_{ij}], \\ \textstyle\sum_{j \in J^t_i} \frac{|e^t_{ij} - \bar{e}^t_{ij}|}{\Delta e^t_{ij}} + \sum_{j \in J^r_i} \frac{|e^r_{ji} - \bar{e}^r_{ji}|}{\Delta e^r_{ji}} \le \Gamma^e_i \}. \end{aligned} \tag{7}$$

We view the uncertainty budget as an ℓ_1-norm constraint for the vector

$$\left(\left(\tfrac{e^t_{ij} - \bar{e}^t_{ij}}{\Delta e^t_{ij}}\right)_{j \in J^t_i}, \left(\frac{e^r_{ji} - \bar{e}^r_{ji}}{\Delta e^r_{ji}}\right)_{j \in J^r_i} \right).$$

Similarly, let $\Gamma^E_i \in [0, 1]$ be the uncertainty budget for E_i, namely $E_i \in [\bar{E}_i - \Gamma^E_i \Delta E_i, \bar{E}_i + \Gamma^E_i \Delta E_i]$. The following *robust* maximum lifetime routing problem is formulated so that we can guarantee feasibility for all data realizations in the restricted uncertainty sets:

$$\max \quad T \tag{8}$$

$$
\begin{aligned}
s.t.\quad & (2),\ (4),\\
& \max_{e^t_{ij}, e^r_{ji} \in \mathscr{R}_i(\Gamma^e_i)} \{ \sum_{j \in \mathscr{S}_i} e^t_{ij} \hat{q}_{ij} + \sum_{j | i \in \mathscr{S}_j} e^r_{ji} \hat{q}_{ji} \} \leq \bar{E}_i - \Gamma^E_i \Delta E_i, \quad \forall i \in \mathscr{N} \setminus \mathscr{D}.
\end{aligned} \tag{9}
$$

In the Appendix we show that the above is equivalent to a linear programming problem.

Theorem 2.1 *The robust problem* (8) *is equivalent to the linear programming formulation:*

$$
\begin{aligned}
\max\quad & T \\
s.t.\quad & \sum_{j \in \mathscr{S}_i} \bar{e}^t_{ij} \hat{q}_{ij} + \sum_{j | i \in \mathscr{S}_j} \bar{e}^r_{ji} \hat{q}_{ji} + \sum_{j \in J^t_i} \omega_{ij} + \sum_{j \in J^r_i} \nu_{ji} \\
& + \Gamma^e_i p_i \leq \bar{E}_i - \Gamma^E_i \Delta E_i, \quad \forall i \in \mathscr{N} \setminus \mathscr{D}, \\
& (2),\ (4),\ p_i \geq 0,\ \forall i \in \mathscr{N} \setminus \mathscr{D}, \\
& p_i + \omega_{ij} \geq \Delta e^t_{ij} \hat{q}_{ij},\ \omega_{ij} \geq 0,\ \forall i \in \mathscr{N} \setminus \mathscr{D}, j \in J^t_i, \\
& p_i + \nu_{ji} \geq \Delta e^r_{ji} \hat{q}_{ji},\ \nu_{ji} \geq 0,\ \forall i \in \mathscr{N} \setminus \mathscr{D}, j \in J^r_i.
\end{aligned} \tag{10}
$$

Furthermore, solving (10) *we obtain an optimal solution* $(\hat{\mathbf{q}}^R, T^R, \mathbf{p}^R, \omega^R, \nu^R)$ *so that* $(\hat{\mathbf{q}}^R, T^R)$ *is feasible for* (8) *and* T^R *is equal to the optimal value of* (8).

2.2 Properties of optimal solutions

Next, we study the relationships between the three formulations and establish properties of the optimal solutions. We also introduce a metric —the *lifetime guarantee probability*— to quantify how likely it is for the predicted lifetime to be achieved.

2.2.1 Optimal lifetime

Let T^*_N, T^*_F, T^*_R denote the optimal values of the nominal, fat, and robust problems, respectively. Let $\mathbf{\Gamma}^e = (\Gamma^e_1, \ldots, \Gamma^e_{|\mathscr{N} \setminus \mathscr{D}|})$ and $\mathbf{\Gamma}^E = (\Gamma^E_1, \ldots, \Gamma^E_{|\mathscr{N} \setminus \mathscr{D}|})$. Note that T^*_R depends on $\mathbf{\Gamma}^e$ and $\mathbf{\Gamma}^E$. To express this dependence, we write $T^*_R(\mathbf{\Gamma}^e, \mathbf{\Gamma}^E)$. The following proposition is almost immediate. It simply states that by adjusting the uncertainty budgets one can generate a continuum of formulations whose predicted lifetime ranges from the fat to the nominal.

Proposition 2.2 $T^*_R(\mathbf{\Gamma}^e, \mathbf{\Gamma}^E)$ *is a non-increasing function of both* $\mathbf{\Gamma}^e$ *and* $\mathbf{\Gamma}^E$. *Furthermore,* $T^*_F \leq T^*_R(\mathbf{\Gamma}^e, \mathbf{\Gamma}^E) \leq T^*_N$.

Proof : Fix $\mathbf{\Gamma}^{e1}, \mathbf{\Gamma}^{e2}, \mathbf{\Gamma}^{E1}, \mathbf{\Gamma}^{E2}$ so that $\mathbf{\Gamma}^{e1} \leq \mathbf{\Gamma}^{e2}$, and $\mathbf{\Gamma}^{E1} \leq \mathbf{\Gamma}^{E2}$. It follows that $\mathscr{R}_i(\Gamma^{e1}_i) \subseteq \mathscr{R}_i(\Gamma^{e2}_i)$, for all $i \in \mathscr{N} \setminus \mathscr{D}$. Let $\hat{\mathbf{q}}^2$ be an optimal flow for the robust routing problem under $\mathbf{\Gamma}^{e2}, \mathbf{\Gamma}^{E2}$. For all $i \in \mathscr{N} \setminus \mathscr{D}$, we have

$$\begin{aligned}
&\max_{e^t_{ij}, e^r_{ji} \in \mathscr{R}_i(\Gamma^{e1}_i)} \{\textstyle\sum_{j \in \mathscr{S}_i} e^t_{ij} \hat{q}^2_{ij} + \sum_{j | i \in \mathscr{S}_j} e^r_{ji} \hat{q}^2_{ji}\} \\
&\leq \max_{e^t_{ij}, e^r_{ji} \in \mathscr{R}_i(\Gamma^{e2}_i)} \{\textstyle\sum_{j \in \mathscr{S}_i} e^t_{ij} \hat{q}^2_{ij} + \sum_{j | i \in \mathscr{S}_j} e^r_{ji} \hat{q}^2_{ji}\} \\
&\leq \bar{E}_i - \Gamma^{E2}_i \Delta E_i \leq \bar{E}_i - \Gamma^{E1}_i \Delta E_i,
\end{aligned}$$

which suggests that $\hat{\mathbf{q}}^2$ is a feasible flow vector for the robust routing problem under $\mathbf{\Gamma}^{e1}, \mathbf{\Gamma}^{E1}$. It follows that $T^*_R(\mathbf{\Gamma}^{e1}, \mathbf{\Gamma}^{E1}) \geq T^*_R(\mathbf{\Gamma}^{e2}, \mathbf{\Gamma}^{E2})$.

Next notice that when $\mathbf{\Gamma}^e = \mathbf{0}$, $\mathbf{\Gamma}^E = \mathbf{0}$, the uncertainty set becomes $\mathscr{R}_i(\Gamma^e_i) = \{e^t_{ij}, e^r_{ji} | e^t_{ij} = \bar{e}^t_{ij}, e^r_{ji} = \bar{e}^r_{ji}\}$ and the robust routing problem (8) reduces to the nominal routing problem (1), that is, $T^*_R(\mathbf{0}, \mathbf{0}) = T^*_N$.

When $\mathbf{\Gamma}^e = (|J^t_1| + |J^r_1|, \ldots, |J^t_{|\mathscr{N} \setminus \mathscr{D}|}| + |J^r_{|\mathscr{N} \setminus \mathscr{D}|}|)$ and $\Gamma^E_i = 1$ for all i the uncertainty sets becomes $\mathscr{R}_i(\Gamma^e_i) = \{e^t_{ij}, e^r_{ji} | e^t_{ij} \in [\bar{e}^t_{ij} - \Delta e^t_{ij}, \bar{e}^t_{ij} + \Delta e^t_{ij}], e^r_{ij} \in [\bar{e}^r_{ij} - \Delta e^r_{ij}, \bar{e}^r_{ij} + \Delta e^r_{ij}]\}$ and $E_i \in [\bar{E}_i - \Delta E_i, \bar{E}_i + \Delta E_i]$ for all i, which implies that the robust routing problem (8) reduces to the fat one (5).

■

Standard sensitivity analysis results from linear programming yield the following corollary.

Corollary 2.3 $T^*_R(\mathbf{\Gamma}^e, \mathbf{\Gamma}^E)$ *is a concave function of* $\mathbf{\Gamma}^E$.

Observe now that at optimality at least one of the energy constraints ((3), (6), (9)) will be active. This is stated in the following proposition. We will call *dead* the nodes that correspond to active constraints at optimality. The lifetime of a dead node equals the lifetime of the network.

Proposition 2.4 *At optimality at least one of the energy constraints in each of the nominal* (1)*, fat* (5)*, and robust* (8) *formulations will be active.*

2.2.2 Optimal flows

Consider an optimal flow vector $\hat{\mathbf{q}}$ obtained by solving one of the three formulations. Recall that $\hat{\mathbf{q}}$ denotes total flow over the lifetime and $\mathbf{q}$ flow per unit of time. We associate a directed graph (subgraph of $\mathscr{G}$) $\mathscr{G}_{\mathbf{q}} = (\mathscr{N}, \mathscr{A}_{\mathbf{q}})$ to $\mathbf{q}$ where $\mathscr{A}_{\mathbf{q}}$ contains all (i,j) with $q_{ij} > 0$. We say that a flow $\mathbf{q}$ is *acyclic (resp., cyclic)* if $\mathscr{G}_{\mathbf{q}}$ contains no cycles (resp., otherwise).

Theorem 2.5 *For all three formulations* (1)*,* (5)*, and* (8)*, there exist acyclic optimal flows.*

Proof : Let $(\mathbf{q}^*, T^*)$ be an optimal solution with $q^*_{i_1 i_2}$, $q^*_{i_2 i_3}$, $q^*_{i_3 i_4}$, $\ldots$, $q^*_{i_k i_1}$ forming a cycle in $\mathscr{G}_{\mathbf{q}^*}$. Let $\Delta q = \min\{q^*_{i_1 i_2}, q^*_{i_2 i_3}, \ldots, q^*_{i_k i_1}\}$. Subtract Δq from all the flows on the cycle. At least one of $q^*_{i_1 i_2}$, $q^*_{i_2 i_3}$, $q^*_{i_3 i_4}$, $\ldots$, $q^*_{i_k i_1}$ becomes zero and all other flows remain non-negative. Because both the in-flow and out-flow at each node is reduced by the same amount, the flow conservation condition for all the nodes $i_1, \ldots, i_k$ still holds. Since the above operation only reduces flows all the energy constraints remain satisfied. Hence, the reduced flows remain optimal. We can repeat the same process to eliminate any other cycle.

■

Since (i, j, i) is a trivial cycle we obtain the following Corollary.

Corollary 2.6 *For all three routing formulations* (1), (5), *and* (8), *there exists an optimal flow* $\mathbf{q}$ *which satisfies* $q_{ij}q_{ji} = 0$ *for all possible links* (i, j) *and* (j, i).

Corollary 2.7 *For all three routing formulations* (1), (5), *and* (8), *there exists an optimal flow* $\mathbf{q}$ *satisfying* $q_{ij} = 0$, $\forall i \in \mathscr{D}$, *which means no flow out of sinks.*

Proof : Let $\mathbf{q}^*$ be an acyclic optimal flow (cf. Thm. 2.5). Suppose there are sinks with positive flows emanating from them. Let $i \in \mathscr{D}$ such that $S_i' \triangleq \{k | q_{ik}^* > 0\}$ is not empty. For $j \in S_i'$ let $S_j' = \{k | q_{jk}^* > 0\}$. We reduce q_{ij}^* to zero by proportionally allocating this flow reduction to all outflows from node j. To be specific, $\forall k_0 \in S_j'$ we set the new reduced flow as $\tilde{q}_{jk_0}^* := q_{jk_0}^* - q_{ij}^* \frac{q_{jk_0}^*}{\sum_{k \in S_j'} q_{jk}^*}$ which maintains the non-negativity of the resulting flow. The flow reduction $q_{ij}^* \frac{q_{jk_0}^*}{\sum_{k \in S_j'} q_{jk}^*}$ can be propagated to the node downstream from j in a similar way. Since $\mathbf{q}^*$ is acyclic and the network is finite, propagating the flow reduction as described above terminates at some other sink nodes. During this process, flow conservation and energy constraints are maintained. This yields a new optimal flow vector with no flows out of sinks.

■

2.2.3 Lifetime guarantee probability

Consider one of the three formulations (1), (5), and (8) and let $\mathbf{q}^*, T^*$ be an optimal solution. We will refer to the probability

$$\mathbf{P}\left[\min_{i \in \mathscr{N} \setminus \mathscr{D}} \frac{E_i}{\sum_{j \in \mathscr{S}_i} e_{ij}^t q_{ij}^* + \sum_{j | i \in \mathscr{S}_j} e_{ji}^r q_{ji}^*} \geq T^*\right],$$

evaluated under the distributions of the r.v.'s E_i, e_{ij}^t, e_{ji}^r, as the *lifetime guarantee probability.* This is the probability that the actual lifetime obtained by applying the optimal flow $\mathbf{q}^*$ achieves the predicted optimal lifetime. We denote by P_N, P_F, P_R the lifetime guarantee probabilities for the nominal (1), fat (5), and robust (8) formulations, respectively. By design, the fat formulation provides an "absolute" guarantee; we omit the proof.

Theorem 2.8 *It holds that* $P_F = 1$.

The straightforward observation is that when $\Gamma_i^e \to |J_i^t| + |J_i^r|$, $\Gamma_i^E \to 1$, $\forall i \in \mathscr{N} \setminus \mathscr{D}$, then $P_R \to P_F$; while when $\Gamma_i^e \to 0$, $\Gamma_i^E \to 0$, $\forall i \in \mathscr{N} \setminus \mathscr{D}$, then $P_R \to P_N$.

Let now $\mathscr{A}^N$ be the set of nodes having active energy constraints at optimality in the nominal formulation (1). For any random variable a with mean $\bar{a}$ and support in $[\bar{a} - \Delta a, \bar{a} + \Delta a]$ we say that it is *symmetrically distributed* if $F_a(\bar{a} - \delta) = 1 - F_a(\bar{a} + \delta)$ for all $\delta \in [0, \Delta a]$, where F_a is the cumulative distribution function of a.

Theorem 2.9 *If E_i, e_{ij}^t, e_{ji}^r are independent symmetrically distributed r.v.'s, then $P_N \leq (\frac{1}{2})^{|\mathscr{A}^N|}$.*

Proof : Let $(\mathbf{q}^{*N}, T_N^*)$ be an optimal solution to the nominal problem (1). We have

$$P_N \leq \mathbf{P}\left[E_i \geq \sum_{j \in \mathscr{S}_i} e_{ij}^t \hat{q}_{ij}^{*N} + \sum_{j | i \in \mathscr{S}_j} e_{ji}^r \hat{q}_{ji}^{*N}, \forall i \in \mathscr{A}^N\right]. \tag{11}$$

For $i \in \mathscr{A}^N$ and because $\mathbf{q}^{*N}$ is feasible for the nominal problem it holds $\bar{E}_i = \sum_{j \in \mathscr{S}_i} \bar{e}_{ij}^t \hat{q}_{ij}^{*N} + \sum_{j | i \in \mathscr{S}_j} \bar{e}_{ji}^r \hat{q}_{ji}^{*N}$. Since E_i, $e_{ij}^t \hat{q}_{ij}^{*N}$, $e_{ji}^r \hat{q}_{ji}^{*N}$ are independent symmetrically distributed r.v.'s with means $\bar{E}_i$, $\bar{e}_{ij}^t \hat{q}_{ij}^{*N}$, $\bar{e}_{ji}^r \hat{q}_{ji}^{*N}$, respectively, it follows that

$$\mathbf{P}[E_i \geq \textstyle\sum_{j \in \mathscr{S}_i} e_{ij}^t \hat{q}_{ij}^{*N} + \sum_{j | i \in \mathscr{S}_j} e_{ji}^r \hat{q}_{ji}^{*N}] = \frac{1}{2}.$$

By independence, we have $P_N \leq (\frac{1}{2})^{|\mathscr{A}^N|}$.

■

2.3 Linear and square arrays

In this section we study two regular network topologies: linear and square arrays. Linear arrays appear, for instance, in pipeline monitoring applications and square arrays are applicable in environmental monitoring applications.

2.3.1 Linear arrays

We consider a *linear array segment* where one sink node is at the center and an equal number k of sensor nodes are aligned one by one on both sides of the sink. The distance between neighboring nodes is d. The radio range is in $[2d, 3d)$, i.e., every node can only communicate with its very next 4 neighbors. Lining up multiple such segments we can build a *linear array network*. We grow the network in this manner since one would need a sink per given number of sensor nodes. We assume that all sensor nodes have identical characteristics, that is, E_i has the same distribution for all i, e_{ij}^t and e_{ij}^r have the same distribution among equidistant nodes, and the information generation rate Q_i is identical for all i. The network we described is motivated by oil or gas pipeline monitoring applications. The following theorem establishes a decomposition property.

Theorem 2.10 *The maximum lifetime routing problem under either the nominal* (1)*, fat* (5)*, or robust formulation* (8) *for a linear array network described above can be decomposed into the corresponding subproblems for each one of its segments.*

Proof : Without loss of generality consider a linear array network denoted by L consisting of two segments L_1 and L_2. Consider any of the three routing formulations and let $T^*_{L_1}$, $T^*_{L_2}$, T^*_L be the optimal values for networks L_1, L_2, and, L respectively. Clearly, $T^*_{L_1} = T^*_{L_2} \le T^*_L$ since by combining the optimal flow vectors for L_1 and L_2 we obtain a feasible flow vector for L.

Due to homogeneity and symmetry in L, there exists an optimal flow vector which is symmetric about the center of L. Flows in the interface between the two segments L_1 and L_2 can fall into one out of two possible cases shown in Fig. 1 (top). In each case, we can reconstruct the optimal flows between nodes k and $k-1$ of L_1 and nodes $-k$ and $-k+1$ of L_2 as shown in 1 (bottom). This flow reconstruction process maintains feasibility and eliminates any communication between segments L_1 and L_2. Then $T^*_L = \min\{T_{L_1}, T_{L_2}\} \le T^*_{L_1} = T^*_{L_2}$. Together with our earlier observation it follows $T^*_L = T^*_{L_1} = T^*_{L_2}$, which establishes the result.

■

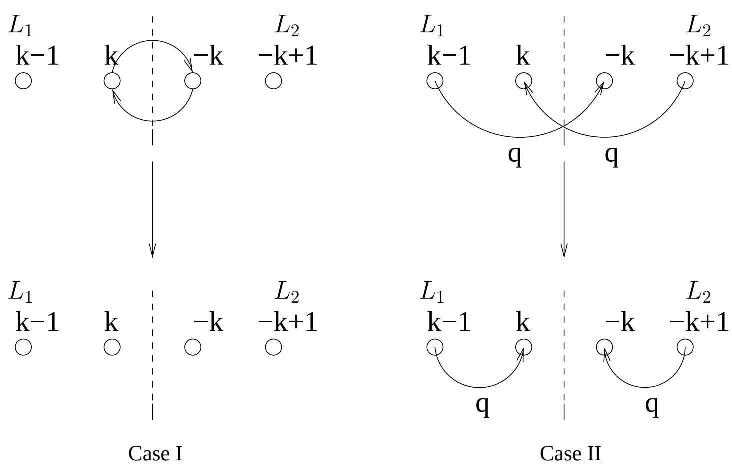


Figure 1: Flow reconstruction for an optimal flow of L.

The following theorem establishes that the nominal formulation (1) is not particularly useful since its predicted lifetime will be achieved with a diminishing probability as the size of the network increases.

Theorem 2.11 *Consider a WSNET formed by aligning 2^n linear arrays as described before. Assume E_i, e^t_{ij}, e^r_{ji} are i.i.d. and non-degenerate r.v.'s (i.e., not equal to a constant). Then, as $n \to \infty$, $P_N \to 0$.*

Proof : By applying Thm. 2.10 n times, we decompose the network L into 2^n identical segments. With this decomposition, we have identical optimal flows in all 2^n linear segments. As we have seen before, each segment has at least one node with a binding energy constraint. Let $\mathscr{K}$ denote a set which contains one node from each segment with a binding energy constraint. It follows that

$$\begin{aligned} P_N &= \prod_{i\in L} \mathbf{P}[E_i \geq \sum_{j\in\mathscr{S}_i} e^t_{ij}\hat{q}^{*N}_{ij} + \sum_{j|i\in\mathscr{S}_j} e^r_{ji}\hat{q}^{*N}_{ji}] \\ &\leq \prod_{k\in\mathscr{K}} \mathbf{P}[E_k \geq \sum_{j\in\mathscr{S}_k} e^t_{kj}\hat{q}^{*N}_{kj} + \sum_{j|k\in\mathscr{S}_j} e^r_{jk}\hat{q}^{*N}_{jk}] \\ &= \prod_{k\in\mathscr{K}} \mathbf{P}[E_k \geq \bar{E}_k], \end{aligned}$$

where the last equality follows from the fact that every $k \in \mathscr{K}$ corresponds to a binding energy constraint. Notice $\mathbf{P}[E_k \geq \bar{E}_k] < 1$ for non-degenerate r.v.'s and that $|\mathscr{K}| = 2^n$. Hence, as $n \to \infty$, $P_N \to 0$.

■

2.3.2 Square arrays

A *square array network* consists of *square array segments*. Each segment is a two-dimensional (square) grid of a given dimension with a node at each point in the grid and a sink node located at the center point of the grid. The vertical and horizontal distance between neighboring nodes is d and we assume that the radio range is slightly less than $\sqrt{5}d$. As with linear arrays, we assume that all sensor nodes have identical characteristics, i.e., E_i has the same distribution for all i, e^t_{ij} and e^r_{ij} have the same distribution among equidistant nodes, and the information generation rate Q_i is identical for all i. We grow a square network in both dimensions by stitching together segments. As an example, a network S with four segments $S_1, \ldots, S_4$ can be formed by placing segment S_1 in the northeast orthant, segment S_2 in the southeast orthant, S_3 in the southwest orthant, and S_4 in the northwest orthant. The following result is analogous to Thm. 2.10.

Theorem 2.12 *Consider a network S consisting of 4 segments $S_1, \ldots, S_4$ as outlined above. The maximum lifetime routing problem under either formulation ((1), (5), or (8)) for S can be decomposed into the corresponding problems for $S_1, \ldots, S_4$.*

Proof : Fix a particular formulation, fat, nominal, or robust. Let $T^*_{S_i}$, T^*_S be the optimal values for network S_i, $i = 1, \ldots, 4$, and S, respectively. As in the proof of Thm. 2.10 $T^*_{S_i} \leq T^*_S$ for all i.

Due to the homogeneity and symmetry of S, there exists an optimal flow vector for S with no flows out of sinks which is symmetric about the vertical line

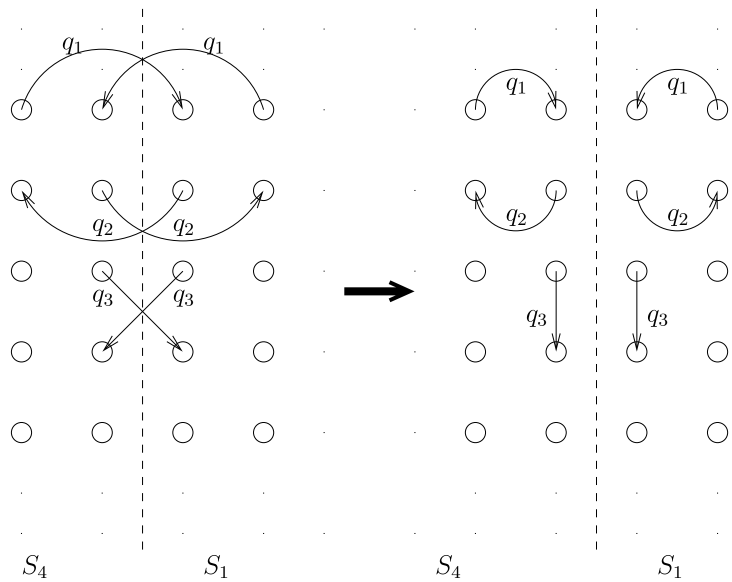


Figure 2: Flow reconstruction for an optimal flow of S.

that separates (S_4, S_3) and (S_1, S_2). As in Thm. 2.10 we consider all possible cases and reconstruct the optimal flow as shown in Fig. 2 (right), resulting in the new flow with no communication between (S_1, S_2) and (S_4, S_3). A similar flow reconstruction process can result in a flow with no communication between (S_1, S_4) and (S_2, S_3). These flow reconstruction steps maintain flow conservation and do not violate the energy constraints, so the resulting flow is optimal. It follows that $T_S^* \leq T_{S_i}^*$ for all i which concludes the proof.

■

Analogous to the linear array case we can now show that the nominal formulation does not provide a useful lifetime prediction. We omit the proof as it is similar to the proof of Thm. 2.11.

Theorem 2.13 *Let a square network be constructed by repeating n times the process of constructing S from $S_1, \ldots, S_4$. Assume E_i, e_{ij}^t, e_{ji}^r are i.i.d. and non-degenerate r.v.'s (i.e., not equal to a constant). Then, as $n \to \infty$, $P_N \to 0$.*

2.4 Uncertainty only in E_i

Here we focus on the case where uncertainty appears only in the initial available energy E_i. Namely, for all results in this subsection we assume that e_{ij}^t's and e_{ji}^r's are known with certainty.

We define a *global robustness budget* $\Gamma = \sum_{\forall i \in \mathscr{N} \setminus \mathscr{D}} \Gamma_i$ and incorporate the allocation of Γ to individual Γ_i into the following robust formulation:

$$\begin{aligned} \max \quad & T && (12)\\ s.t. \quad & \textstyle\sum_{j \in \mathscr{S}_i} e^t_{ij} \hat{q}_{ij} + \sum_{j | i \in \mathscr{S}_j} e^r_{ji} \hat{q}_{ji} \leq \bar{E}_i - \Gamma_i \Delta E_i, \ \forall i \in \mathscr{N} \setminus \mathscr{D}, \\ & (2),\ (4), \\ & \textstyle\sum_{\forall i \in \mathscr{N} \setminus \mathscr{D}} \Gamma_i = \Gamma,\ 0 \leq \Gamma_i \leq 1,\ \forall i \in \mathscr{N} \setminus \mathscr{D}, && (13) \end{aligned}$$

where the decision variables are T, the $\hat{q}_{ij}$'s, and the Γ_i's. The following monotonicity property is immediate. Concavity follows from the fact that (12) maximizes a concave (linear) objective over linear constraints and Γ appears in the right hand side of these constraints.

Proposition 2.14 *The optimal value T^*_R of* (12) *is monotonically non-increasing and concave as a function of the global robustness budget Γ.*

2.4.1 Optimizing $\mathbf{P}[T \geq T^*]$ over the optimal flows $\mathbf{q}^*$

When the uncertainty is only in E_i's, we can maximize the lifetime guarantee probability $\mathbf{P}[T \geq T^*]$ over the set of optimal flows $\mathbf{q}^*$ while guaranteeing that we achieve the corresponding predicted lifetime. One can think of this optimization as maximizing "robustness" while guaranteeing the same objective (predicted lifetime). We next show that this problem is a well-structured concave optimization problem. We only treat the robust case. For the fat case we have already shown that $P_F = 1$ and the nominal case is similar to the robust.

Assume that only E_i's are uncertain and let T^*_R, $\mathbf{s}^*$, $\mathbf{q}^*, \Gamma^*$ form an optimal solution of the robust formulation (12), where $\mathbf{s}^*$ denotes the vector of slack variables corresponding to the energy constraints. Suppose all E_i's are independent; then

$$\begin{aligned} P_R =& \mathbf{P}\left[\min_{i \in \mathscr{N} \setminus \mathscr{D}} \frac{E_i}{\sum_{j \in \mathscr{S}_i} \bar{e}^t_{ij} q^*_{ij} + \sum_{j | i \in \mathscr{S}_j} \bar{e}^r_{ji} q^*_{ji}} \geq T^*_R \right] \\ =& \textstyle\prod_{i \in \mathscr{N} \setminus \mathscr{D}} \mathbf{P}[E_i \geq \bar{E}_i - s^*_i - \Gamma^*_i \Delta E_i]. \end{aligned}$$

Taking the E_i's to be uniformly distributed in $[\bar{E}_i - \Delta E_i, \bar{E}_i + \Delta E_i]$:

$$\mathbf{P}[E_i \geq \bar{E}_i - s^*_i - \Gamma^*_i \Delta E_i] = \frac{\Delta E_i + \min\{s^*_i + \Gamma^*_i \Delta E_i, \Delta E_i\}}{2 \Delta E_i} \triangleq p_i,$$

and $P_R = \prod_{i \in \mathscr{N} \setminus \mathscr{D}} p_i$ where we defined $2 p_i \Delta E_i = \Delta E_i + \min\{s^*_i + \Gamma^*_i \Delta E_i, \Delta E_i\}$.

To maximize P_R while achieving the optimal lifetime T^*_R, we can equivalently maximize $\ln(P_R)$ which yields the following concave optimization problem:

$$\begin{aligned} & \max\ \ln(P_R) = \textstyle\sum_{i \in \mathscr{N} \setminus \mathscr{D}} \ln p_i && (14)\\ & s.t.\ (2),\ \textstyle\sum_{j \in \mathscr{S}_i} \bar{e}^t_{ij} \hat{q}_{ij} + \sum_{j | i \in \mathscr{S}_j} \bar{e}^r_{ji} \hat{q}_{ji} + s_i = \bar{E}_i - \Gamma_i \Delta E_i,\ \forall i \in \mathscr{N} \setminus \mathscr{D}, \\ & \quad T \geq T^*_R,\ s_i \geq 0,\ \hat{q}_{ij} \geq 0,\ \forall i \in \mathscr{N},\ \forall j \in \mathscr{S}_i, \end{aligned}$$

$2p_i \Delta E_i - \Delta E_i \leq s_i + \Gamma_i \Delta E_i, \ \forall i \in \mathscr{N} \setminus \mathscr{D},$
(13), $2p_i \Delta E_i - \Delta E_i \leq \Delta E_i, \ \forall i \in \mathscr{N} \setminus \mathscr{D}.$

3 Maximum lifetime routing with energy allocation

In this section, we consider the problem of maximizing the WSNET lifetime by jointly optimizing the routing decisions and the initial energy allocated to the nodes. Suppose E is the total available energy for a WSNET. Similar to formulation (1) we have the *nominal* problem:

$$\begin{aligned} \max \quad & T \\ s.t. \quad & (2),\ (3),\ (4), \\ & \textstyle\sum_{i \in \mathscr{N} \setminus \mathscr{D}} E_i = E,\ E_i \geq 0, \quad \forall i \in \mathscr{N} \setminus \mathscr{D}. \end{aligned} \tag{15}$$

Here the E_i's (appearing in (3) and above) are decision variables. The corresponding fat and robust formulations respectively are

$$\begin{aligned} \max \quad & T \\ s.t. \quad & \textstyle\sum_{j \in \mathscr{S}_i} \bar{e}^t_{ij} \hat{q}_{ij} + \sum_{j \in J^t_i} \Delta e^t_{ij} \hat{q}_{ij} + \sum_{j | i \in \mathscr{S}_j} \bar{e}^r_{ji} \hat{q}_{ji} + \\ & \qquad \textstyle\sum_{j \in J^r_i} \Delta e^r_{ji} \hat{q}_{ji} \leq E_i, \quad \forall i \in \mathscr{N} \setminus \mathscr{D}, \\ & (2),\ (4), \\ & \textstyle\sum_{i \in \mathscr{N} \setminus \mathscr{D}} E_i = E,\ E_i \geq 0, \quad \forall i \in \mathscr{N} \setminus \mathscr{D}, \quad \text{and} \end{aligned} \tag{16}$$

$$\max \quad T \tag{17}$$

$$s.t. \quad \max_{e^t_{ij}, e^r_{ji} \in \mathscr{R}_i(\Gamma^e_i)} \left\{ \textstyle\sum_{j \in \mathscr{S}_i} e^t_{ij} \hat{q}_{ij} + \sum_{j | i \in \mathscr{S}_j} e^r_{ji} \hat{q}_{ji} \right\} \leq E_i, \quad \forall i \in \mathscr{N} \setminus \mathscr{D}, \tag{18}$$

$$\begin{aligned} & (2),\ (4), \\ & \textstyle\sum_{i \in \mathscr{N} \setminus \mathscr{D}} E_i = E,\ E_i \geq 0, \quad \forall i \in \mathscr{N} \setminus \mathscr{D}. \end{aligned}$$

As before, the robust problem (17) can be shown to be equivalent to a linear programming problem; we omit the details for brevity. From the structure of the formulation with energy allocation, we have the following result.

Proposition 3.1 *At optimality, all the energy constraints for non-sink nodes are active and the total energy constraint is also binding. This holds for all three formulations.*

Proof : Consider first the robust problem (17). We will use contradiction. Assume that at optimality the energy constraint (18) for some non-sink node k is not active. Notice that we can decrease E_k and increase all the other E_i while maintaining their sum. This improves the lifetime which contradicts optimality. Similarly, the total energy constraint is also binding at optimality. If not, we can increase all E_i to achieve a better lifetime, which again contradicts optimality. The nominal and fat cases are almost identical.

■

3.1 Properties of optimal solutions

As before, let T_N^*, T_F^*, T_R^* denote the optimal values of the nominal, fat, and robust routing problems, respectively. Let $\mathbf{\Gamma}^e = (\Gamma_1^e, \ldots, \Gamma_{|\mathscr{N} \setminus \mathscr{D}|}^e)$. Note that T_R^* depends on $\mathbf{\Gamma}^e$. To express this dependence, we write $T_R^*(\mathbf{\Gamma}^e)$. The following result is similar to Prop. 2.2.

Proposition 3.2 *$T_R^*(\mathbf{\Gamma}^e)$ is a non-increasing function of $\mathbf{\Gamma}^e$ and $T_F^* \leq T_R^*(\mathbf{\Gamma}^e) \leq T_N^*$.*

As in Sec. 2.2, we associate a directed graph $\mathscr{G}_{\mathbf{q}} = (\mathscr{N}, \mathscr{A}_{\mathbf{q}})$ to a feasible flow vector $\mathbf{q}$ where $\mathscr{A}_{\mathbf{q}}$ contains all (i,j) with $q_{i,j} > 0$. Recall that we name $\mathbf{q}$ as *acyclic* when $\mathscr{G}_{\mathbf{q}}$ contains no cycles. The following results are similar to Prop. 2.5 and Corollary 2.7; we omit the proofs.

Proposition 3.3 *For all three formulations* (15)*,* (16)*, and* (17)*, the optimal flows are acyclic.*

Proposition 3.4 *For all three formulations the optimal flows $\mathbf{q}$ satisfy $q_{ij} = 0$, $\forall i \in \mathscr{D}$.*

3.2 Lifetime guarantee probability

The development in this section is similar to Section 2. We have the following results; we omit the details in the interest of brevity.

Proposition 3.5 *It holds $P_F = 1$.*

Note that when $\Gamma_i^e \to |J_i^t| + |J_i^r|$, $\forall i \in \mathscr{N} \setminus \mathscr{D}$, then $P_R \to P_F$; while when $\Gamma_i^e \to 0$, $\forall i \in \mathscr{N} \setminus \mathscr{D}$, then $P_R \to P_N$.

Proposition 3.6 *If e_{ij}^t,e_{ji}^r are independent symmetrically distributed r.v.'s, then $P_N = (\frac{1}{2})^{|\mathscr{N} \setminus \mathscr{D}|}$.*

It follows that as $|\mathscr{N} \setminus \mathscr{D}| \to \infty$ we have $P_N \to 0$ and this now holds for all topologies.

4 Routing and energy allocation in massively dense WSNETs

It is straightforward that the joint problem of routing and energy allocation (15) is equivalent to finding paths from sources to sinks with lowest energy consumption rate. If we consider the energy consumed by both the sender and the receiver over a link as the cost (or length) of the link, the problem is reduced to finding shortest paths between sources and sinks. Imagine now that the WSNET is scaled by uniformly deploying an increasing number of nodes while decreasing their radio range in order to maintain a fixed density of one-hop-reachable neighbors. Although the approach we developed so far scales well since we are dealing with linear programming problems, it is of interest to consider whether the scaled problem exhibits, in the limit, a structure that simplifies its solution and deepens our understanding. In particular, we will consider a *limiting regime* of massively dense WSNETs and study maximum lifetime routing formulations with energy allocation. Such WSNETs can only be described by *macroscopic* parameters, such as the information generation and energy distribution densities.

4.1 Problem formulation

Let $\mathscr{M}$ be the planar area where a massively dense WSNET is deployed. Mathematically, $\mathscr{M}$ is a convex set in $\mathbb{R}^2$. We assume that the WSNET is uniformly deployed over $\mathscr{M}$.

Let $Q(x, y)$ represent the *information generation density function* defined on $\mathscr{M}$ whose units are bits/(sec $\cdot$ m^2). We assume $Q(x, y)$ is known. Denote by $S(x, y)$ the *information consumption density function* defined on $\mathscr{M}$ whose units are bits/(sec $\cdot$ m^2). In the next subsection we will consider the special cases of "point" sources and sinks where $Q(x, y)$ and $S(x, y)$ become Dirac functions on the plane. Let $e(x, y)$ be the *energy density function* defined on $\mathscr{M}$ whose units are J/m^2. The energy density function $e(x, y)$ characterizes the distribution of the globally available energy E over $\mathscr{M}$. Define the *information traffic flow function* as $\mathbf{q}(x, y) = (q_x(x, y), q_y(x, y))$. The interpretation of $\mathbf{q}(x, y)$ is as follows: $\epsilon\|\mathbf{q}(x, y)\|$ is the rate at which information crosses a linear segment of infinitesimal length ϵ which is centered on (x, y) and perpendicular to $\mathbf{q}(x, y)$ (see Fig. 3). The units of $\|\mathbf{q}\|$ are bits/(sec $\cdot$ m).

The continuous maximum life routing problem with energy allocation can be formulated as:

$$\max \quad T \tag{19}$$

$$s.t. \quad \frac{\partial q_x(x,y)}{\partial x} + \frac{\partial q_y(x,y)}{\partial y} = Q(x, y) - S(x, y),\ \forall (x, y) \in \mathscr{M}, \tag{20}$$

$$T \leq \lim_{\epsilon \to 0} \frac{\int_{\Omega(\epsilon)} e(x,y) \mathrm{d}\sigma}{\alpha\ \epsilon\ \|\mathbf{q}(x,y)\|},\ \forall (x, y) \in \mathscr{M}, \tag{21}$$

$$\int_{\mathscr{M}} e(x, y) \mathrm{d}\sigma = E, \tag{22}$$

$$\int_{\mathscr{M}} (Q(x, y) - S(x, y)) \mathrm{d}\sigma = 0, \tag{23}$$

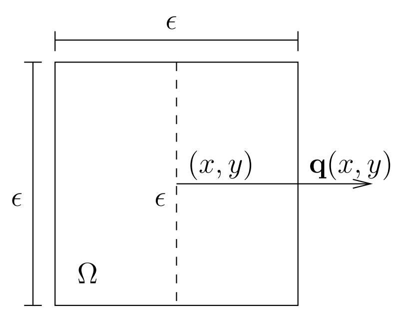


Figure 3: The information traffic flow function $\mathbf{q}(x, y)$.

$$T \geq 0,\ S(x,y) \geq 0,\ e(x,y) \geq 0, \quad \forall (x,y) \in \mathscr{M},$$

where $S(x, y)$, $e(x, y)$, $\mathbf{q}(x, y)$, and T are decision functions and variables. Using an argument in [TT06], (20) states that the divergence of the traffic flow function measures the degree with which the traffic increases or decreases; we can think of this as a detailed flow conservation equation. (22) is a global energy constraint while (23) can be seen as a global flow conservation constraint. As for (21), consider a point $(x, y) \in \mathscr{M}$ and let $\Omega(\epsilon)$ denote an infinitesimal square centered at (x, y) with a side length equal to ϵ and one of its sides parallel to $\mathbf{q}(x, y)$. Let α (in J/(bit · sec)) be a constant indicating how much energy is consumed per unit of transmitted information per second. Then, (21) expresses the fact that the total energy consumed when the traffic flow $\mathbf{q}(x, y)$ passes through $\Omega(\epsilon)$ during a period of time T should be no more than the total energy available in this area.

In this section we are only interested in the structure of the optimal solutions to (19), hence we only consider the nominal version of the problem. Uncertainty in E can be easily incorporated as we have done with the discrete instances. This will only change the right hand side of the total energy constraint and would not affect the optimal solution structure. Uncertainty in $e(x, y)$ can also be incorporated but that is beyond the main focus of this section.

From the structure of (19), we have the following results. The proof is immediate as whenever $\|\mathbf{q}(x, y)\| = 0$ and $e(x, y) > 0$ we can reduce $e(x, y)$ to zero while maintaining feasibility. The energy savings can be allocated to other points resulting in a potential increase of the lifetime.

Proposition 4.1 (19) *has optimal solutions such that* $e(x, y) = 0$ *whenever* $\|\mathbf{q}(x, y)\| = 0$.

Similar to Prop. 3.1 we can show:

Proposition 4.2 *For problem* (19)*, there exist optimal solutions such that the detailed energy constraints* (21) *are all active.*

4.2 Single point source and sink

In this subsection, we focus on the scenario where there is a single point source and a single point sink in a massively dense WSNET. We start with the definition of a point source/sink.

Definition 1 (Point Source on (x_o, y_o))
Let $\mathbf{o} = (x_o, y_o)$ *be the location of the source on* $\mathscr{M}$ *and denote by* Q *its information generation rate. The point information generation density function* $Q_{\mathbf{o}}(x, y)$ *satisfies*

$$Q_{\mathbf{o}}(x,y) = \begin{cases} 0 & (x,y) \neq (x_o,y_o), \\ +\infty & (x,y) = (x_o,y_o), \end{cases} \qquad \textit{and} \qquad \int_{\mathbb{R}^2} Q_{\mathbf{o}}(x,y)\mathrm{d}\sigma = Q.$$

Similarly, we define the information consumption density function $S_{\mathbf{s}}(x, y)$ for a point sink at (x_s, y_s) with a sink rate equal to S. These are Dirac impulse functions on $\mathbb{R}^2$.

In the single point source and single point sink case, let $\mathbf{o} = (x_o, y_o)$ and $\mathbf{s} = (x_s, y_s)$ be the source and sink locations, respectively. Denote by $Q_o(x, y)$ and $S_s(x, y)$ the corresponding information generation/consumption density function with rates Q and S, respectively. It follows from the global flow balance equation (23) that $Q = S$. We next define the notion of a marginal density function; its units are J/m.

Definition 2 (Marginal Energy Density function on Curve C)
Let C *be a continuous curve connecting two points* $\mathbf{o}$ *and* $\mathbf{s}$*, and let* $\mathscr{T}(C, \epsilon)$ *denote an* ϵ*-tube around* C *as shown in Fig. 4. The marginal energy density function* $e_C(x, y)$ *on curve* C *satisfies*

$$e_C(x,y) = \begin{cases} 0, & (x,y) \notin C, \\ \lim_{\epsilon\to 0} \frac{\int_{\Omega(\epsilon)} e(x,y)\mathrm{d}\sigma}{\epsilon}, & (x,y) \in C. \end{cases} \tag{24}$$

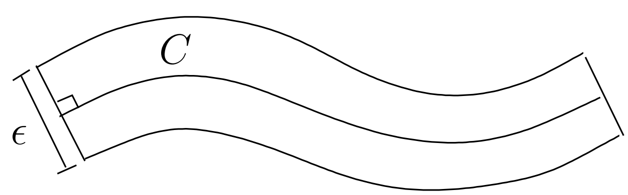


Figure 4: C and its ϵ-tube.

4.2.1 Properties of optimal solutions

Let $\mathbf{o} = (x_o, y_o)$ and $\mathbf{s} = (x_s, y_s)$ be the source and sink positions, respectively. Information generated at the source (with rate Q) gets consumed at the sink (with rate S, where it follows that $Q = S$).

Consider an arbitrary set of paths (see Fig. 5) traversed by the traffic as it flows from $\mathbf{o}$ to $\mathbf{s}$. Specifically, the traffic flow first follows curve C_o then forks

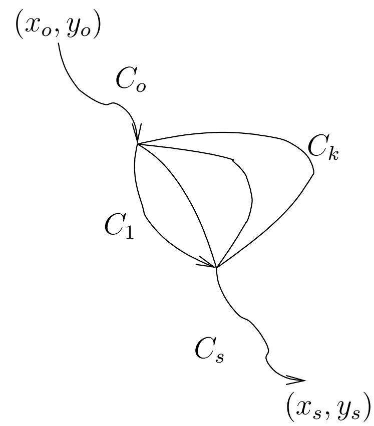


Figure 5: An arbitrary set of paths from $\mathbf{o}$ to $\mathbf{s}$.

into branches $C_1, C_2, \ldots, C_k$ which merge into C_s. Denote by $Q_o, \ldots, Q_k, Q_s$ the traffic flows on $C_o, \ldots, C_k, C_s$, respectively, where $\sum_{i=1}^k Q_i = Q_o = Q_s$. We have already established that at optimality we have an energy density function that is nonzero only on the curves $Q_o, \ldots, Q_k, Q_s$ and all energy constraints are active. The problem reduces to

$$\begin{aligned} \max \quad & T \\ s.t. \quad & \alpha T Q_i = e_{C_i}, \quad i \in \{o, 1, 2, \ldots, k, s\} \\ & \textstyle\sum_i e_{C_i} \ell(C_i) = E, \ T \geq 0, \ e_{C_i} \geq 0, \quad \forall (x, y) \in C_i, \end{aligned}$$

where $\ell(C_i)$ denotes the length of the curve C_i. Note this is a linear program. Given the allocation of Q into $Q_1, \ldots, Q_k$ an optimal lifetime is

$$T = \frac{E}{\sum_{i \in \{o,1,2,\ldots,k,s\}} \alpha Q_i \ell(C_i)}.$$

It follows that to maximize T, the branches $C_o, \ldots, C_k, C_s$ should all be straight lines (minimal length). Furthermore, the best T can be achieved by a straight line from $\mathbf{o}$ to $\mathbf{s}$. The result is summarized below.

Proposition 4.3 *The path that maximizes the network's lifetime is the straight line from* $\mathbf{o}$ *to* $\mathbf{s}$. *The corresponding energy distribution function* $e(x, y)$ *is nonzero only on this line with a uniform marginal energy density function.*

We note that the argument above can be extended to handle an infinite number of (forked and merged) paths. The key idea is the same, i.e., we can show that any solution using an infinite number of paths is no better than the straight line connecting $\mathbf{o}$ with $\mathbf{s}$. We will omit the details to avoid obfuscating the discussion.

4.3 Multiple point sources and sinks

The result in the previous subsection readily generalizes to the situation where we have n point sources, say $o_1, \ldots, o_n$, and m sinks, denoted by $s_1, \ldots, s_m$. The result is provided in the following proposition; we omit the details because it follows the same line of reasoning.

Proposition 4.4 *For problem* (19) *with multiple point sources and sinks, there exist an optimal solution such that every source sends its information to its nearest sink along the straight line segment connecting them and the corresponding marginal energy density function on the line segment is uniform.*

The result implies that sinks generate a Voronoi tessellation of the deployment area and the sources send their flows over straight lines to the sink in the cell they reside in, thus, resulting in a star-like network within each cell.

5 Numerical experiments

In this section we present a set of numerical examples. For all examples we adopt the *communication energy consumption* model from [CT04].

Let d^r be the transmission range of each node. Then $j \in \mathscr{S}_i$ if and only if $d_{ij} \leq d^r$, where d_{ij} is the distance between nodes i and j. The energy expenditure per data unit transmitted from i to j satisfies $e^t_{ij} = e^\circ + \epsilon_{amp} d^4_{ij}$, $e^r_{ij} = e^R$, where $e^\circ = 50$ nJ/bit and $e^R = 150$ nJ/bit denote the energy consumed in the transceiver circuitry at the transmitter and the receiver, respectively, and $\epsilon_{amp} = 100$ pJ/bit/m^4 is the energy consumed at the output transmitter antenna for transmitting a bit over one meter. The receiver circuitry is in general more complex and consumes more energy than the transmitter circuitry within the same order of magnitude. The path loss exponent of four is chosen to account for multipath reflections. In all the numerical experiments P_R is estimated by Monte-Carlo simulation with 10^6 samples, thus P_R is accurate with a ± 0.005 error and 99% confidence (by Chebyshev's inequality).

5.1 A 4-node WSNET

We start with a toy example to give some intuition on the routing policies produced by each formulation. The WSNET consists of one origin node O, two relay nodes, R_1 and R_2, and one sink node S, where $Q_O = 500$ bits/sec and the radio range is 30 m. The origin node O has to use relays R_1 and R_2 to reach the sink S. Further, 20m $= d_{OR_1} = d_{R_1S} < d_{OR_2} = d_{R_2S} =$ 21m. First we consider the case without energy allocation.

5.1.1 Routing without energy allocation

All E_i, e^t_{ij}, e^r_{ji} are uniformly distributed with $E_O \in [9.0, 10.0]$J where $\bar{E}_O =$ 9.5J, $E_{R_1} \in [8.1, 11.1]$J where $\bar{E}_{R_1} =$ 9.6J, $E_{R_2} \in [9.5, 10.5]$J with $\bar{E}_{R_2} =$ 10.0J, $\Delta e^t_{OR_1}/\bar{e}^t_{OR_1} = \Delta e^t_{R_1S}/\bar{e}^t_{R_1S} = 0.35$, and $\Delta e^t_{ij}/\bar{e}^t_{ij} = 0.1$ for all other appropriate i and j. $\Delta e^r_{ji}/\bar{e}^r_{ji} = 0.1$ for all appropriate i and j. Note that $\Gamma^e_O \in [0, 4]$, $\Gamma^e_{R_1} \in [0, 6]$, $\Gamma^e_{R_2} \in [0, 6]$, and $\Gamma^E_i \in [0, 1]$ for all i. Take $\Gamma^e_O = 1.04$, $\Gamma^e_{R_1} = \Gamma^e_{R_2} = 1.56$, and $\Gamma^E_i = 0.26$ for all i.

In Fig. 6(a), the red (dot-dash), black (dash), and green (solid-star) lines with arrows represent the nominal, fat, and robust optimal flows, respectively. Note the difference in the selected routes: the nominal picks the shorter path $O - R_1 - S$, the fat picks the more "stable" but a little longer path $O - R_2 - S$, while the robust balances the two to maintain a relatively high lifetime guarantee probability while not suffering too much from the low predicted lifetime.

As we adjust $\frac{\Gamma^e_i}{|J^t_i|+|J^r_i|} = \Gamma^E_i$, P_R and T^*_R will change accordingly. The solid blue curve in Fig. 6(b) describes the relationships between P_R and $\frac{T^*_R - T^*_F}{T^*_F}$ (the percentage predicted lifetime gain of the robust formulation over the fat). It can be seen that there is significant predicted lifetime gain (e.g., 15%) while the lifetime guarantee probability remains high (e.g., close to 0.8). The red dash curve represents the relationship $\frac{\Gamma^e_i}{|J^t_i|+|J^r_i|} = \Gamma^E_i$ v.s. P_R. It can be seen that as we protect more against the randomness, the predicted lifetime T^*_R goes down and the lifetime guarantee probability P_R gets enhanced. The two extreme cases of no protection and full protection correspond to the nominal and fat situations.

To gain further insight on the impact of uncertainty on the nominal formulation consider the probability distribution of the actual lifetime T achieved by applying the nominal optimal policy $\mathbf{q}^*_N$ to random instances (where e^t_{ij}, e^r_{ji}, and E_i are randomly selected). Fig. 7 shows the histogram of T generated from a million instances. We can see that T can be substantially smaller than T^*_N and in fact most of the probability mass corresponds to such T's. The nominal lifetime guarantee probability $P_N = \mathbf{P}[T \geq T^*_N]$ would be fairly low but that does not capture how far from T^*_N the actual lifetime T can be.

5.1.2 Routing with energy allocation

If energy allocation is an option, set the global available energy $E = 30$J. As before, $\Delta e^t_{OR_1}/\bar{e}^t_{OR_1} = \Delta e^t_{R_1S}/\bar{e}^t_{R_1S} = 0.35$, and $\frac{\Delta e^t_{ij}}{\bar{e}^t_{ij}} = 0.1$ for all other appropriate i and j. $\Delta e^r_{ji}/\bar{e}^r_{ji} = 0.1$ for all appropriate i and j. Note that $\Gamma^e_O \in [0,4]$, $\Gamma^e_{R_1} \in [0,6]$, $\Gamma^e_{R_2} \in [0,6]$. Take $\Gamma^e_O = 0.8$, $\Gamma^e_{R_1} = \Gamma^e_{R_2} = 1.2$.

Table 1: T^* and lifetime guarantee probability for the 4-node WSNET.

No Energy Allocation	Nominal	Fat	Robust
T^* ($\frac{T^*_R - T^*_F}{T^*_F} \times 100\%$)	1183.8	839.24	931.64 (11%)
Lifetime guarantee prob.	0.25	1.0	0.90
With Energy Allocation	Nominal	Fat	Robust
T^* ($\frac{T^*_R - T^*_F}{T^*_F} \times 100\%$)	1860.47	1393.38	1448.4 (4%)
Lifetime guarantee prob.	0.25	1.0	0.82

Fig. 8(a) presents the nominal, fat, and robust optimal flows and energy allocation. The situation is very similar as before but energy allocation improves the predicted lifetime since no energy is wasted. Optimal values in a number of nominal, fat, and robust cases with and without energy allocation are listed in Table 1.

5.2 A randomly deployed WSNET

In this case, we have 20 nodes (4 sinks, 10 origins, 6 relays) uniformly deployed on a 50×50 m^2 square. $d^r = 25$m. $Q_i = 500$ bits/sec, $\forall i \in \mathscr{O}$. All E_i, e^t_{ij}, e^r_{ji} are uniformly distributed and $\bar{E}_i = 10$J, $\Delta E_i/\bar{E}_i$ is uniformly sampled from $[0, 0.3]$. $J^t_i = \mathscr{S}_i$, $J^r_i = \{j | i \in \mathscr{S}_j\}$, $\Delta e^t_{ij}/\bar{e}^t_{ij}$ and $\Delta e^r_{ji}/\bar{e}^r_{ji}$ are uniformly sampled from $[0, 0.4]$.

5.2.1 Routing without energy allocation

We use $\frac{\Gamma^e_i}{|J^t_i|+|J^r_i|} = \Gamma^E_i = 0.71$ and solve the three routing problems without energy allocation. The policies are quite different since we compute $\frac{||\mathbf{q}^*_N - \mathbf{q}^*_F||}{||\mathbf{q}^*_F||} = 0.33$ and $\frac{||\mathbf{q}^*_R - \mathbf{q}^*_F||}{||\mathbf{q}^*_F||} = 0.92$.

5.2.2 Routing with energy allocation

Let now $\frac{\Gamma^e_i}{|J^t_i|+|J^r_i|} = 0.09$. Solving the problems with energy allocation we obtain $\frac{||\mathbf{q}^*_N - \mathbf{q}^*_F||}{||\mathbf{q}^*_F||} = 0.32$ and $\frac{||\mathbf{q}^*_R - \mathbf{q}^*_F||}{||\mathbf{q}^*_F||} = 0.19$. The results for both cases are presented in Table 2.

Table 2: T^* and lifetime guarantee probability for the randomly deployed WSNET.

<table>
<tr><td colspan="4">Routing without energy allocation for the Random WSNETs</td></tr>
<tr><td></td><td>Nominal</td><td>Fat</td><td>Robust</td></tr>
<tr><td>T^* ($\frac{T_R^*-T_F^*}{T_F^*} \times 100\%$)</td><td>2249.84</td><td>1151.67</td><td>1287.74 (11.8%)</td></tr>
<tr><td>Lifetime guarantee prob.</td><td>0.063</td><td>1.0</td><td>0.905</td></tr>
<tr><td colspan="4">Routing without energy allocation for the Random WSNETs</td></tr>
<tr><td>T^* ($\frac{T_R^*-T_F^*}{T_F^*} \times 100\%$)</td><td>19039.9</td><td>15693.8</td><td>16056.2 (2.31%)</td></tr>
<tr><td>Lifetime guarantee prob.</td><td>5.09×10^{-4}</td><td>1.0</td><td>0.89</td></tr>
</table>

Again adjusting $\frac{\Gamma_i^e}{|J_i^t|+|J_i^r|} = \Gamma_i^E$ or $\frac{\Gamma_i^e}{|J_i^t|+|J_i^r|}$, respectively for the two cases, changes P_R and T_R^* accordingly (see Figs. 9(a) and 9(b)). It can be seen that as we protect more against the randomness, the predicted lifetime T_R^* goes down and the lifetime guarantee probability P_R gets enhanced. For energy allocation problems, since at optimality all energy constraints are active, the lifetime guarantee probability gets reduced but still the gain over the fat formulation is non-negligible.

As we did in the 4-node example we plot in Fig. 10 the histogram of T achieved by $\mathbf{q}_N^*$ computed from a million random instances of the problem (without energy allocation). It is clear that as the number of nodes grows the probability mass for T shifts away from T_N^* and the actual T is typically substantially smaller than T_N^*. This is consistent with our result that $P_N = \mathbf{P}[T \geq T_N^*] \to 0$.

6 Conclusions

We presented a new framework to accommodate uncertainty in designing maximum lifetime routing policies for WSNETs. We considered two scenarios — one (Scenario A) assuming that energy is already allocated to various nodes and the other (Scenario B) where such allocation is also subject to optimization. We formulated a worst case (fat) problem and compared it with the nominal problem that makes certainty equivalence assumptions and ignores uncertainty. As a compromise between the two we also devised a robust formulation. We established, analytically and numerically, that the nominal solutions are always too optimistic. Specifically, for common Scenario A topologies (like regular linear arrays and grid-like WSNETs) the nominal formulation predicts a lifetime that is (almost) never achieved in the presence of uncertainty. In Scenario B, the sane result holds for all topologies. The robust solutions, on the other hand, provide a useful and practical way to trade-off performance vs. robustness. We extended our analysis to massively dense WSNETs and characterized optimal solutions of the routing problems.

Appendix

A Proof of Theorem 2.1

Let $(\hat{\mathbf{q}}^*, T^*, \mathbf{p}^*, \omega^*, \nu^*)$ and $(\hat{\mathbf{q}}^\circ, T^\circ)$ be optimal solutions of (10) and (8), respectively. We will show that $(\hat{\mathbf{q}}^*, T^*)$ is a feasible solution of (8) with $T^* = T^\circ$. For any $\hat{\mathbf{q}} \geq \mathbf{0}$, the maximization problem in the energy constraint for node i in the robust problem (8) is

$$\max_{e^t_{ij}, e^r_{ji} \in \mathscr{R}_i(\Gamma^e_i)} \{\textstyle\sum_{j \in \mathscr{S}_i} e^t_{ij}\hat{q}_{ij} + \sum_{j: i \in \mathscr{S}_j} e^r_{ji}\hat{q}_{ji}\}. \tag{25}$$

Note that (25) is equivalent to the following linear optimization problem:

$$\begin{aligned} \max \quad & \textstyle\sum_{j \in \mathscr{S}_i} \bar{e}^t_{ij}\hat{q}_{ij} + \sum_{j \in J^t_i} \Delta e^t_{ij}\hat{q}_{ij}z_{ij} + \sum_{j | i \in \mathscr{S}_j} \bar{e}^r_{ji}\hat{q}_{ji} + \sum_{j \in J^r_i} \Delta e^r_{ji}\hat{q}_{ji}\lambda_{ji} \qquad (26) \\ s.t. \quad & \textstyle\sum_{j \in J^t_i} z_{ij} + \sum_{j \in J^r_i} \lambda_{ji} \leq \Gamma^e_i, \\ & 0 \leq z_{ij} \leq 1, \ \forall j \in J^t_i, \\ & 0 \leq \lambda_{ji} \leq 1, \ \forall j \in J^r_i, \end{aligned}$$

where (z_{ij}, λ_{ji}) are the decision variables. Then the dual of (26) is:

$$\begin{aligned} \min \quad & \textstyle\sum_{j \in \mathscr{S}_i} \bar{e}^t_{ij}\hat{q}_{ij} + \sum_{j | i \in \mathscr{S}_j} \bar{e}^r_{ji}\hat{q}_{ji} + \Gamma^e_i p_i + \sum_{j \in J^t_i} \omega_{ij} + \sum_{j \in J^r_i} \nu_{ji} \qquad (27) \\ s.t. \quad & p_i + \omega_{ij} \geq \Delta e^t_{ij}\hat{q}_{ij}, \ \forall j \in J^t_i, \\ & p_i + \nu_{ji} \geq \Delta e^r_{ji}\hat{q}_{ji}, \ \forall j \in J^r_i, \\ & \omega_{ij} \geq 0, \ \forall j \in J^t_i, \ \nu_{ji} \geq 0, \ \forall j \in J^r_i, \ p_i \geq 0, \end{aligned}$$

where $(p_i, \omega_{ij}, \nu_{ji})$ are the dual variables. Fix $\hat{\mathbf{q}} = \hat{\mathbf{q}}^*$ in (26) and (27), and let $(z^*_{ij}, \lambda^*_{ji})$ be an optimal solution of (26). Note that $(p^*_i, \omega^*_{ij}, \nu^*_{ji})$ is feasible for (27). For all $i \in \mathscr{N} \setminus \mathscr{D}$ we have

$$\begin{aligned} & \max_{e^t_{ij}, e^r_{ji} \in \mathscr{R}_i(\Gamma^e_i)} \{\textstyle\sum_{j \in \mathscr{S}_i} e^t_{ij}\hat{q}^*_{ij} + \sum_{j: i \in \mathscr{S}_j} e^r_{ji}\hat{q}^*_{ji}\} \\ & = \textstyle\sum_{j \in \mathscr{S}_i} \bar{e}^t_{ij}\hat{q}^*_{ij} + \sum_{j \in J^t_i} \Delta e^t_{ij}\hat{q}^*_{ij}z^*_{ij} + \sum_{j | i \in \mathscr{S}_j} \bar{e}^r_{ji}\hat{q}^*_{ji} + \sum_{j \in J^r_i} \Delta e^r_{ji}\hat{q}^*_{ji}\lambda^*_{ji} \\ & \leq \textstyle\sum_{j \in \mathscr{S}_i} \bar{e}^t_{ij}\hat{q}^*_{ij} + \sum_{j | i \in \mathscr{S}_j} \bar{e}^r_{ji}\hat{q}^*_{ji} + \Gamma^e_i p^*_i + \sum_{j \in J^t_i} \omega^*_{ij} + \sum_{j \in J^r_i} \nu^*_{ji} \\ & \leq \bar{E}_i - \Gamma^E_i \Delta E_i, \end{aligned}$$

where the first equation is due to the equivalence of (25) and (26), the following inequality is due to the weak duality between (26) and (27), and the second inequality is due to the feasibility of $(\hat{\mathbf{q}}^*, T^*, \mathbf{p}^*, \omega^*, \nu^*)$ in (10). This shows that $(\hat{\mathbf{q}}^*, T^*)$ is feasible to (8), implying that $T^* \leq T^\circ$.

Next, set $\hat{\mathbf{q}} = \hat{\mathbf{q}}^\circ$ in (26) and (27), and let $(z^\circ_{ij}, \lambda^\circ_{ji})$ be an optimal solution to (26). By strong duality, there exists a feasible $(p^\circ_i, \omega^\circ_{ij}, \nu^\circ_{ji})$ to (27) such that for all $i \in \mathscr{N} \setminus \mathscr{D}$,

$$\bar{E}_i - \Gamma^E_i \Delta E_i \geq \max_{e^t_{ij}, e^r_{ji} \in \mathscr{R}_i(\Gamma^e_i)} \{\textstyle\sum_{j \in \mathscr{S}_i} e^t_{ij}\hat{q}^\circ_{ij} + \sum_{j: i \in \mathscr{S}_j} e^r_{ji}\hat{q}^\circ_{ji}\}$$

$$= \sum_{j\in\mathscr{S}_i} \bar{e}^t_{ij}\hat{q}^\circ_{ij} + \sum_{j\in J^t_i} \Delta e^t_{ij}\hat{q}^\circ_{ij}z^\circ_{ij} + \sum_{j|i\in\mathscr{S}_j} \bar{e}^r_{ji}\hat{q}^\circ_{ji} + \sum_{j\in J^r_i} \Delta e^r_{ji}\hat{q}^\circ_{ji}\lambda^\circ_{ji}$$
$$= \sum_{j\in\mathscr{S}_i} \bar{e}^t_{ij}\hat{q}^\circ_{ij} + \sum_{j|i\in\mathscr{S}_j} \bar{e}^r_{ji}\hat{q}^\circ_{ji} + \Gamma^e_i p^\circ_i + \sum_{j\in J^t_i} \omega^\circ_{ij} + \sum_{j\in J^r_i} \nu^\circ_{ji}.$$

Thus, $(\hat{\mathbf{q}}^\circ, T^\circ, \mathbf{p}^\circ, \omega^\circ, \nu^\circ)$ satisfies the second set of constraints of (10). Since the remaining constraints are also satisfied, $(\hat{\mathbf{q}}^\circ, T^\circ, \mathbf{p}^\circ, \omega^\circ, \nu^\circ)$ is a feasible solution of (10), hence, $T^\circ \leq T^*$.

References

[ALR03] J. Aslam, Q. Li, and D. Rus, *Three power-aware routing algorithms for sensor networks*, Wireless Communications and Mobile Computing **3** (2003), no. 2, 187–208.

[BE81] D. Baker and A. Ephremides, *The Architectural Organization of a Mobile Radio Network via a Distributed Algorithm*, Communications, IEEE Transactions on [legacy, pre-1988] **29** (1981), no. 11, 1694–1701.

[BS04] D. Bertsimas and M. Sim, *The price of robustness*, Operations Research **52** (2004), no. 1, 35–53.

[BTN99] A. Ben-Tal and A. Nemirovski, *Robust solutions of uncertain linear programs*, Operations Research Letters **25** (1999), 1–13.

[CT04] J.H. Chang and L. Tassiulas, *Maximum lifetime routing in wireless sensor networks*, IEEE/ACM Trans. Networking **12** (2004), no. 4, 609–619.

[GCNB99] J. Gomez, AT Campbell, M. Naghshineh, and C. Bisdikian, *Power-aware routing in wireless packet networks*, Mobile Multimedia Communications, 1999.(MoMuC'99) 1999 IEEE International Workshop on (1999), 380–383.

[GK05] A. Giridhar and PR Kumar, *Maximizing the functional lifetime of sensor networks*, Information Processing in Sensor Networks, 2005. IPSN 2005. Fourth International Symposium on (2005), 5–12.

[Jac04] P. Jacquet, *Geometry of information propagation in massively dense ad hoc networks*, Proceedings of the 5th ACM international symposium on Mobile ad hoc networking and computing (2004), 157–162.

[LH01] L. Li and JY Halpern, *Minimum-energy mobile wireless networks revisited*, Communications, 2001. ICC 2001. IEEE International Conference on **1** (2001).

[LHB+01] L. Li, J.Y. Halpern, P. Bahl, Y.M. Wang, and R. Wattenhofer, *Analysis of a cone-based distributed topology control algorithm for wireless multi-hop networks*, Proceedings of the twentieth annual ACM symposium on Principles of distributed computing (2001), 264–273.

[LP08] W. Lai and I. Ch. Paschalidis, *Optimally balancing energy consumption versus latency in sensor network routing*, ACM Trans. Sensor Networks **4** (2008), no. 4, 21:1–21:28.

[ME98] A. Michail and A. Ephremides, *A distributed routing algorithm for supporting connection-oriented service in wireless networks with time-varying connectivity*, Proceedings of the 3rd IEEE International Symposium on Communications and Control (1998).

[ML06] R. Madan and S. Lall, *Distributed algorithms for maximum lifetime routing in wireless sensor networks*, IEEE Transactions on Wireless Communications **5** (2006), no. 8, 2185–2193.

[NM05] H. Nama and N. Mandayam, *Sensor networks over information fields: optimal energy and node distributions*, Wireless Communications and Networking Conference, 2005 IEEE **3** (2005).

[PK05] I. Ch. Paschalidis and S.-C. Kang, *Robust linear optimization: On the benefits of distributional information and applications in inventory control*, Proceedings of the 44th IEEE Conference on Decision and Control (Seville, Spain), December 2005, pp. 4416–4421.

[RLP06] S. Ray, W. Lai, and I. Ch. Paschalidis, *Statistical location detection with sensor networks*, Joint special issue IEEE/ACM Trans. Networking and IEEE Trans. Information Theory **52** (2006), no. 6, 2670–2683.

[RM99] V. Rodoplu and TH Meng, *Minimum energy mobile wireless networks*, Selected Areas in Communications, IEEE Journal on **17** (1999), no. 8, 1333–1344.

[SHC+04] V. Shnayder, M. Hempstead, B. Chen, G.W. Allen, and M. Welsh, *Simulating the power consumption of large-scale sensor network applications*, Proceedings of the 2nd international conference on Embedded networked sensor systems, 2004, pp. 188–200.

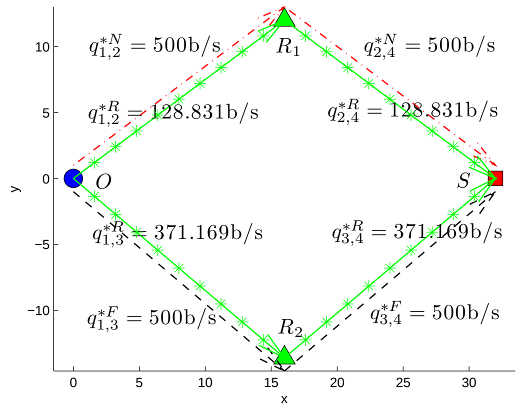
[Soy73] A.L. Soyster, *Convex programming with set-inclusive constraints and applications to inexact linear programming*, Operations Research **21** (1973), 1154–1157.

[TCS01] C. Toh, H. Cobb, and D. Scott, *Performance Evaluation of Battery-Life-Aware Routing Optimization Schemes for Wireless Ad Hoc Networks*, Proc. IEEE Int'l Computing Conf.(ICC'01) (2001).

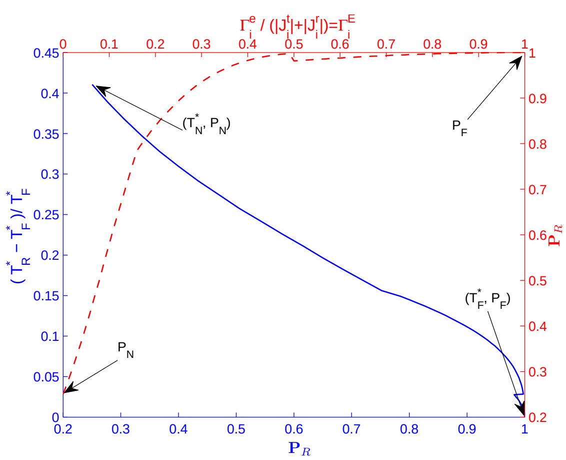
[TT06] S. Toumpis and L. Tassiulas, *Optimal deployment of large wireless sensor networks*, IEEE Transactions on Information Theory **52** (2006), no. 7, 2935–2953.

[WC05] X. Wu and CG Cassandras, *A maximum time optimal control approach to routing in sensor networks*, 44th IEEE Conference on Decision and Control, 2005, pp. 1137–1142.

[WLBW01] R. Wattenhofer, L. Li, P. Bahl, and Y.M. Wang, *Distributed topology control for power efficient operation in multihop wireless ad hoc networks*, INFOCOM 2001. Twentieth Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings. IEEE **3** (2001).



(a)



(b)

Figure 6: (a) Optimal Flows. (b) P_R v.s. $\frac{T_R^* - T_F^*}{T_F^*}$ (solid blue) and $\frac{\Gamma_i^e}{|J_i^t|+|J_i^r|} = \Gamma_i^E$ v.s. P_R (dash red).

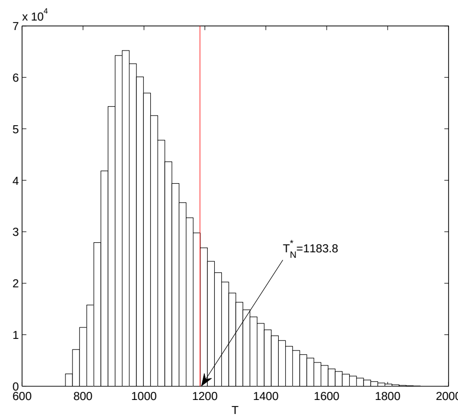
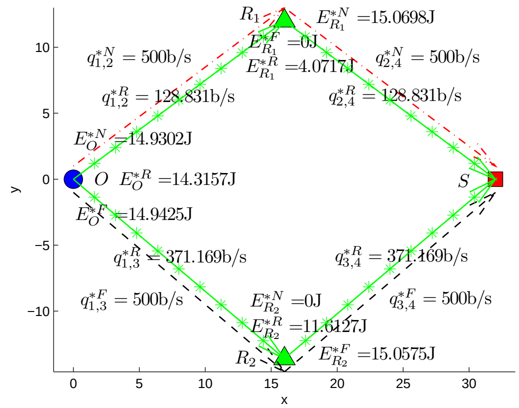
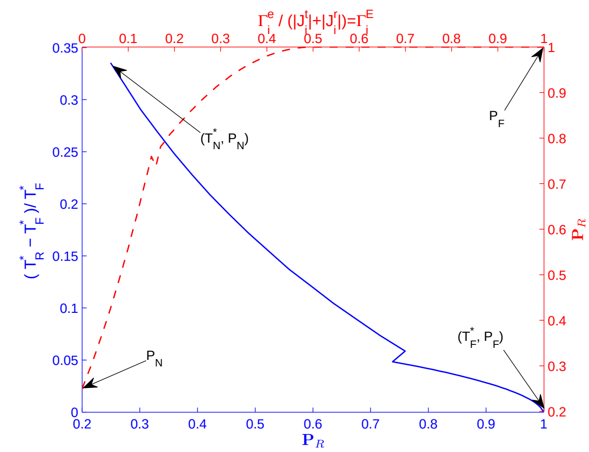


Figure 7: Histogram of T under $\mathbf{q}_N^*$ for the 4node WSNET.

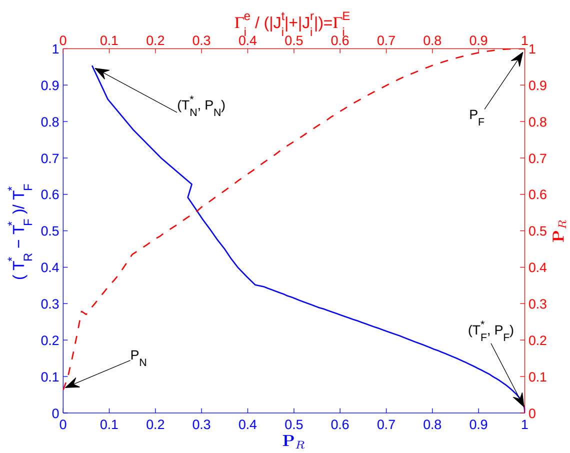


(a)

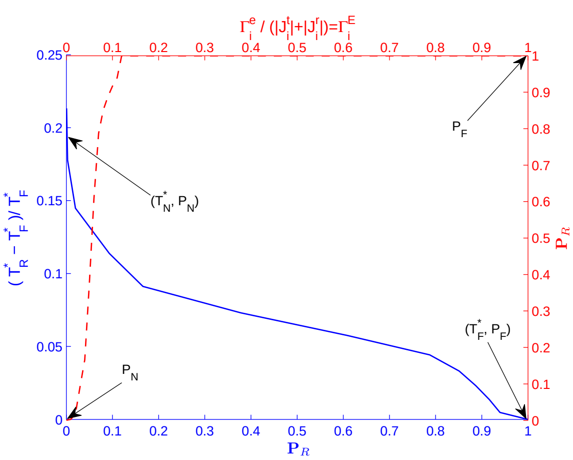


(b)

Figure 8: (a) Optimal Flows and Energy. The red (dot-dash), black (dash), and green (solid-star) lines depict the nominal, fat, and robust optimal flows, respectively. (b) P_R v.s. $\frac{T_R^* - T_F^*}{T_F^*}$ (solid blue) and $\frac{\Gamma_i^e}{|J_i^t|+|J_i^r|} = \Gamma_i^E$ v.s. P_R (dash red).



(a)



(b)

Figure 9: (a) P_R v.s. $\frac{T_R^* - T_F^*}{T_F^*}$ (blue-solid) and $\frac{\Gamma_i^e}{|J_i^t|+|J_i^r|} = \Gamma_i^E$ v.s. P_R (red-dash) without energy allocation. (b) P_R v.s. $\frac{T_R^* - T_F^*}{T_F^*}$ (blue-solid) and $\frac{\Gamma_i^e}{|J_i^t|+|J_i^r|} = \Gamma_i^E$ v.s. P_R (red-dash) with energy allocation.

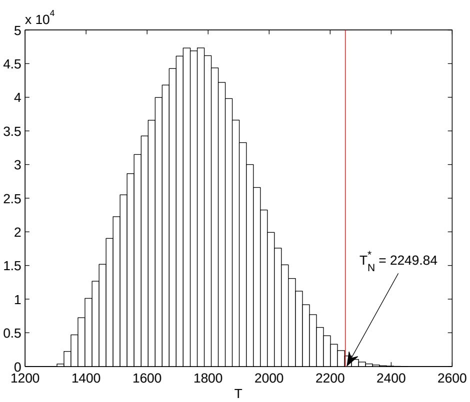


Figure 10: Histogram of T under $\mathbf{q}_N^*$ for the 20 node randomly deployed WSNET.