Confidence intervals

\((1-\alpha)\cdot 100\%\) confidence interval for a parameter \(r:\)

\[ P[A \leq r \leq B] > 1-\alpha \]

We can be \((1-\alpha)\cdot 100\%\) confident that \(r\) lies between \(A\) and \(B\).

Confidence interval for the mean (known variance)

\(X_1, X_2, ..., X_n = N\) i.i.d samples of RV \(X\)

\[ M_n = \frac{1}{n} \sum_{i=1}^{n} X_i \]

\[ E[M_n] = \mu_x \]

\[ Var(M_n) = \frac{\sigma^2}{n} \]

Central limit theorem:

\[ Z_n = \frac{M_n - E[M_n]}{\sqrt{Var[M_n]}} = \frac{M_n - \mu_x}{\frac{\sigma}{\sqrt{n}}} \quad \rightarrow \quad Z_n \sim N(0,1) \quad \text{for large } n \]

\[ P[Z > z_{1/2}] = Q(z_{1/2}) = \frac{\alpha}{2} \]

\[ P[-z_{1/2} \leq Z \leq z_{1/2}] = 1-\alpha \]

\[ P[-z_{1/2} \leq \frac{M_n - \mu_x}{\frac{\sigma}{\sqrt{n}}} \leq z_{1/2}] = 1-\alpha \quad \text{(1-\alpha)\cdot 100\% confidence interval.} \]

\[ M_n - \frac{\sigma}{\sqrt{n}} \leq \mu_x \leq M_n + \frac{\sigma}{\sqrt{n}} \cdot z_{1/2} \]
1. Measurements of the diameters of a random sample of 200 ball bearings made by a certain machine during one week showed a mean of 0.824 inch. The standard deviation is 0.042 inch. Construct a 95\% two-sided confidence interval for the mean diameter of all the ball bearings.

Solution:
Xi = diameter of ball bearing i
\[ M_n = \frac{1}{n} \sum_{i=1}^{n} X_i = 0.824 \]
\[ 6x = 0.042 \]

Since \( n = 200 \) is large, we can assume that \( M_n \) is very nearly normal.

Thus, we can use the formula:

\[ M_n - \frac{6x}{\sqrt{n}} Z_{\alpha/2} \leq M_x \leq M_n + \frac{6x}{\sqrt{n}} Z_{\alpha/2} \]

Step 1: find \( \alpha \)
95\% \( \Rightarrow \) 100(1-\( \alpha \))\% = 95\% \( \Rightarrow \) \( \alpha = 0.05 \)

Step 2: find \( Z_{\alpha/2} \)
\[ Z_n = \frac{M_n - M_x}{6x/\sqrt{n}} \xrightarrow{n \to \infty} Z = N(0,1) \text{ by central limit theorem.} \]
\[ p(Z > Z_{\alpha/2}) = 1 - \Phi(Z_{\alpha/2}) = \frac{\alpha}{2} \]
\[ \Rightarrow \Phi(Z_{\alpha/2}) = 1 - \frac{\alpha}{2} \Rightarrow Z_{\alpha/2} \]

In our case, \( \Phi(Z_{\alpha/2}) = 1 - \frac{0.05}{2} = 0.975 \)
\[ Z_{\alpha/2} = 1.96 \]
\[ \Rightarrow \quad \mu_x = \frac{\bar{x}}{\sqrt{n}} z_{\alpha/2} \leq \mu_x \leq \mu + \frac{\bar{x}}{\sqrt{n}} z_{\alpha/2} \]

\[ 0.824 - \frac{0.042}{\sqrt{200}} \cdot 1.96 \leq \mu_x \leq 0.824 + \frac{0.042}{\sqrt{200}} \cdot 1.96 \]

\[ 0.8182 \leq \mu_x \leq 0.8298 \]

\[ \text{HW10: 2) A sample of 150 brand A light bulbs showed a mean lifetime of 1400 hours. The standard deviation of brand A lifetime is 120 hours. A sample of brand B light bulbs showed a mean lifetime of 1200 hours. The standard deviation of brand B is 80 hours. Construct a 99\% confidence interval on the difference of the two means } MA - MB. \]

**Solution:**

Ai - lifetime of bulb i of brand A
Bi - lifetime of bulb i of brand B

\[ MA = \frac{1}{n} \sum_{i=1}^{n} Ai \]
\[ MB = \frac{1}{m} \sum_{i=1}^{m} Bi \]

\[ E[MA] = \frac{1}{n} \sum_{i=1}^{n} E[Ai] = MA, \quad E[MB] = MB \]

\[ \text{Var}[MA] = \text{Var}\left[\frac{1}{n} \sum_{i=1}^{n} Ai\right] = \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}[Ai] = \frac{\sigma_A^2}{n} \]
\[ \text{Var}[MB] = \frac{\sigma_B^2}{m} \]
\[ m = E[M_A - M_B] = E[M_A] - E[M_B] = m_a - m_b \]

\[ G^2 = \text{Var}[M_A - M_B] = \text{Var}[M_A] + \text{Var}[M_B] = \frac{\sigma_a^2}{n} + \frac{\sigma_b^2}{m} \]

For large \(n, m\), \( Z = \frac{m_a - m_b - m}{\sqrt{n + m}} \sim N(0,1) \)

\[ \Rightarrow M_A - M_B - 6 \alpha \leq m \leq M_A - M_B + 6 \alpha \]

\[ M_A - M_B - \sqrt{\frac{\sigma_a^2}{n} + \frac{\sigma_b^2}{m}} \leq M_A - M_B \leq M_A - M_B + \sqrt{\frac{\sigma_a^2}{n} + \frac{\sigma_b^2}{m}} \cdot 2\alpha \]

General formula for confidence interval on difference of two means when standard deviations \(\sigma_A, \sigma_B\) are known.

In our case,

\[ 99\% \Rightarrow 100(1-\alpha)\% = 99\% \Rightarrow \alpha = 0.01 \]

\[ \Phi(2\alpha) = \frac{1 - \alpha}{2} = 0.995 \Rightarrow Z_{2\alpha/2} = 2.58 \]

\[ n = 150 \]
\[ m = 200 \]

\[ M_A = 1400 \]
\[ M_B = 1200 \]

\[ \sigma_A = 120, \sigma_B = 80 \]

\[ 1400 - 1200 + \sqrt{\frac{120^2}{150} + \frac{80^2}{200}} \cdot 2.58 \leq M_A - M_B \leq 1400 - 1200 + \sqrt{\frac{120^2}{150} + \frac{80^2}{200}} \cdot 2.58 \]

\[ 170.8106 \leq M_A - M_B \leq 229.1894 \]