Recitation 12

May 9, 2008

Problem 1: [YG] Problem 9.2.7

Solution:

We are told that random variable X has a second order Erlang distribution

$$f_X(x) = \begin{cases} \lambda x e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We also know that given $X = x$, random variable Y is uniform on $[0, x]$ so that

$$f_{Y|X}(y|x) = \begin{cases} 1/x & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(a) Given $X = x$, Y is uniform on $[0, x]$. Hence $E[Y|X = x] = x/2$. Thus the minimum mean square estimate of Y given X is

$$\hat{Y}_M(X) = E[Y|X] = X/2 \tag{3}$$

(b) The minimum mean square estimate of X given Y can be found by finding the conditional probability density function of X given Y. First we find the joint density function.

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) \cdot f_X(x) = \begin{cases} \lambda e^{-\lambda x} & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Now we can find the marginal of Y

$$f_Y(y) = \int_y^\infty \lambda e^{-\lambda x}\, dx = \begin{cases} e^{-\lambda y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

By dividing the joint density by the marginal density of Y we arrive at the conditional density of X given Y.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \begin{cases} \lambda e^{-\lambda(x-y)} & x \geq y \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Now we are in a position to find the minimum mean square estimate of X given Y. Given $Y = y$, the conditional expected value of X is

$$E[X|Y = y] = \int_y^\infty \lambda x e^{-\lambda(x-y)}\, dx \tag{7}$$

Making the substitution $u = x - y$ yields

$$E[X|Y = y] = \int_0^\infty \lambda(u + y) e^{-\lambda u}\, du \tag{8}$$

We observe that if U is an exponential random variable with parameter λ, then

$$E[X|Y=y] = E[U+y] = \frac{1}{\lambda} + y \tag{9}$$

The minimum mean square error estimate of X given Y is

$$\hat{X}_M(Y) = E[X|Y] = \frac{1}{\lambda} + Y \tag{10}$$

(c) Since the MMSE estimate of Y given X is the linear estimate $\hat{Y}_M(X) = X/2$, the optimal linear estimate of Y given X must also be the MMSE estimate. That is, $\hat{Y}_L(X) = X/2$.

(d) Since the MMSE estimate of X given Y is the linear estimate $\hat{X}_M(Y) = Y + 1/\lambda$, the optimal linear estimate of X given Y must also be the MMSE estimate. That is, $\hat{X}_L(Y) = Y + 1/\lambda$.

Problem 2: [YG] Problem 8.2.4

Reminder:

minE(cost) : Likelihood ratio $= \frac{f_{Y|H_1}(y|H_1)}{f_{Y|H_0}(y|H_0)} \geq \frac{C_{10}-C_{00}}{C_{01}-C_{11}} \cdot \frac{P[H_0]}{P[H_1]} \Rightarrow$ decide H_1. Otherwise decide H_0.

MAP $(C_{01} = C_{10} = 1, C_{00} = C_{11} = 0)$: $\frac{f_{Y|H_1}(y|H_1)}{f_{Y|H_0}(y|H_0)} \geq \frac{P[H_0]}{P[H_1]} \Rightarrow$ decide H_1. Otherwise decide H_0.

ML (MAP, and $p_0 = p_1$) : $\frac{f_{Y|H_1}(y|H_1)}{f_{Y|H_0}(y|H_0)} \geq 1 \Rightarrow$ decide H_1. Otherwise decide H_0.

Solution:

(a) Given H_0, X is Gaussian (0,1). Given H_1, X is Gaussian (4,1). The likelihood ratio is:

$$\frac{f_{Y|H_1}(y|H_1)}{f_{Y|H_0}(y|H_0)} = \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-4)^2}{2}}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{(x)^2}{2}}} = \frac{e^{-\frac{(x-4)^2}{2}}}{e^{-\frac{(x)^2}{2}}}$$

The MAP hypothesis test is:

$$\frac{e^{-\frac{(x-4)^2}{2}}}{e^{-\frac{x^2}{2}}} \geq \frac{P[H_0]}{P[H_1]} \Rightarrow \text{decide } H_1. \text{ Otherwise decide } H_0.$$

$$\Leftrightarrow -\frac{(x-4)^2}{2} + \frac{x^2}{2} \geq ln\frac{P[H_0]}{P[H_1]}$$

$$\Leftrightarrow x \geq 2 + \frac{1}{4}ln\frac{P[H_0]}{P[H_1]} = 3.15$$

$x \geq 3.15 = x_{MAP} \Rightarrow$ decide H_1. Otherwise decide H_0.

The false alarm and miss probabilities are

$$P_{\text{FA}} = P[X \geq x_{\text{MAP}}|H_0] = Q(x_{\text{MAP}}) = 8.16 \times 10^{-4} \tag{3}$$

$$P_{\text{MISS}} = P[X < x_{\text{MAP}}|H_1] = \Phi(x_{\text{MAP}} - 4) = 1 - \Phi(0.85) = 0.1977. \tag{4}$$

The average cost of the MAP policy is

$$E[C_{\text{MAP}}] = C_{10}P_{\text{FA}}P[H_0] + C_{01}P_{\text{MISS}}P[H_1] \tag{5}$$

$$= (1)(8.16 \times 10^{-4})(0.99) + (10^4)(0.1977)(0.01) = 19.77. \tag{6}$$

(b) The cost of a false alarm is $C_{10} = 1$ unit while the cost of a miss is $C_{01} = 10^4$ units. The Minimum Cost test is the same as the MAP test except the $P[H_0]$ is replaced by $C_{10}P[H_0]$ and $P[H_1]$ is replaced by $C_{01}P[H_1]$. Thus the minimum cost test is:

$$\frac{e^{-\frac{(x-4)^2}{2}}}{e^{-\frac{(x)^2}{2}}} \geq \frac{C_{10}P[H_0]}{C_{01}P[H_1]} \Rightarrow \text{decide } H_1. \text{ Otherwise decide } H_0.$$

$$\Leftrightarrow x \geq 2 + \frac{1}{4} ln \frac{C_{10}P[H_0]}{C_{01}P[H_1]} = 0.846$$

$$x \geq 0.846 = x_{MC} \Rightarrow \text{decide } H_1. \text{ Otherwise decide } H_0.$$

The false alarm and miss probabilities are

$$P_{\text{FA}} = P[X \geq x_{\text{MC}}|H_0] = Q(x_{\text{MC}}) = 0.1987 \tag{8}$$

$$P_{\text{MISS}} = P[X < x_{\text{MC}}|H_1] = \Phi(x_{\text{MC}} - 4) = 1 - \Phi(3.154) = 8.06 \times 10^{-4}. \tag{9}$$

The average cost of the minimum cost policy is

$$E[C_{\text{MC}}] = C_{10}P_{\text{FA}}P[H_0] + C_{01}P_{\text{MISS}}P[H_1] \tag{10}$$

$$= (1)(0.1987)(0.99) + (10^4)(8.06 \times 10^{-4})(0.01) = 0.2773. \tag{11}$$

Because the cost of a miss is so high, the minimum cost test greatly reduces the miss probability, resulting in a much lower average cost than the MAP test.

Problem 4: from 505

The outcome of a random experiment is known to have an exponential distribution, but the parameter α of that distribution is not known. We estimate the parameter α from the sample mean:

$$\widehat{\alpha} = \frac{1}{\widehat{m}} \qquad \widehat{m} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

where x_i are independent trials of the experiment.

(a) Note that $\widehat{m}$ is itself a random variable. Find the mean and variance of $\widehat{m}$.

(b) Use the Chebychev bound to estimate the minimum number of experiments N that are required to guarantee that

$$P[|\widehat{m} - m| > .01m] \leq .001$$

where $m = 1/\alpha$.

(c) Use the Central Limit Theorem to approximate the distribution of the random variable $\widehat{m} - m$ as a Gaussian random variable. Using the Gaussian approximation, estimate the minimum number of experiments N which would be required to guarantee that

$$P[|\widehat{m} - m| > .01m] \leq .001$$

Solution:

Solution:

(a) The mean is given by:

$$E[\widehat{m}] = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i] = \frac{1}{\alpha}$$

The variance is given by:

$$\sigma^2_{\widehat{m}} = \text{Var}\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N^2}\text{Var}\left[\sum_{i=1}^{N} x_i\right] = \frac{1}{N^2}\sum_{i=1}^{N}\text{Var}[x_i] = \frac{1}{N}\frac{1}{\alpha^2}$$

where we have used the fact that $\text{Var}[cX] = c^2\text{Var}[X]$ and the variance of the sum of iid random variables is the sum of the variances of the random variables.

(b) The Chebychev bound states that for a random variable Z with mean m_Z and variance σ_Z^2:

$$\Pr[|Z - m_Z| \geq a] \leq \sigma_Z^2/a^2$$

Now to apply this statement to $P[|\widehat{m} - m| > .01m]$ we would use $a = .01m$, $m_Z = m$, and $\sigma_Z = \sigma_{\widehat{m}}$ found above. This yields:

$$P[|\widehat{m} - m| > .01m] \leq \frac{\sigma^2_{\widehat{m}}}{.0001m^2} = \frac{\alpha^2}{.0001N\alpha^2} = \frac{1}{.0001N}$$

In order for $1/(.0001N) = .001$, we need $\boxed{N = 10^7}$.

(c) The Central Limit Theorem states that for any iid random variables x_i with mean m and variance σ^2 the quantity:

$$\frac{\sum_{i=1}^{N}(x_i - m)}{\sigma\sqrt{N}}$$

approaches a $N(0,1)$ random variable. Note carefully that this variance is the variance of each sample, *not* the variance of the estimator $\sigma^2_{\widehat{m}}$ which you found for part (a). Now we want to make a statement about the quantity:

$$(\widehat{m} - m) = \frac{1}{N}\sum_{i=1}^{N} x_i - m = \frac{1}{N}\sum_{i=1}^{N}(x_i - m) = \frac{\sigma\sqrt{N}}{N}\frac{\sum_{i=1}^{N}(x_i - m)}{\sigma\sqrt{N}} = \frac{\sigma}{\sqrt{N}}\underbrace{\left[\frac{\sum_{i=1}^{N}(x_i - m)}{\sigma\sqrt{N}}\right]}_{Z}$$

Now from the central limit theorem we know that the term in brackets in the last expression, i.e. Z, approaches a $N(0,1)$ random variable. Now with a bit of algebra we get:

$$\begin{aligned}
\Pr\{|\widehat{m} - m| > .01m\} &= \Pr\left\{\frac{\sigma}{\sqrt{N}}\left|\frac{\sum_{i=1}^{N}(x_i - m)}{\sigma\sqrt{N}}\right| > .01m\right\} = \Pr\left\{\left|\frac{\sum_{i=1}^{N}(x_i - m)}{\sigma\sqrt{N}}\right| > \frac{\sqrt{N}}{\sigma}.01m\right\} \\
&= \Pr\left\{\left|\frac{\sum_{i=1}^{N}(x_i - m)}{\sigma\sqrt{N}}\right| > .01\sqrt{N}\right\} \\
&= \Pr\left\{|Z| > .01\sqrt{N}\right\}
\end{aligned}$$

where $Z \sim N(0,1)$ and we have used the fact that for an exponential random variable $m = 1/\alpha$ and $\sigma = 1/\alpha$. For such a normalized Gaussian random variable Z with zero mean and unit variance:

$$Q(\beta) \equiv \Pr(Z > \beta) = \int_{\beta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{y^2/2}\, dy$$

which is tabulated in Appendix D of Shanmugan & Breipohl or in many other places. Thus in terms of the "Q" function, we want:

$$\Pr\{|\widehat{m} - m| > .01m\} = \Pr\left\{|Z| > .01\sqrt{N}\right\} = 2Q\left(.01\sqrt{N}\right) \leq .001$$

where because of the absolute value we multiply $Q(\cdot)$ by two. Now going to the tables we find $Q(3.3) = .001/2 = .0005$. Thus

$$\begin{aligned}
.01\sqrt{N} &= 3.3 \\
N &= 108900
\end{aligned}$$

Notice that this number is much smaller than that found using the bound. This is becausing we are only using an approximation and not a bound. Often bounds lead to overly conservative answers.