Recitation 4

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Problem 1

A random variable X has the desity function $f(x) = c/(x^2 + 1)$, where $-\infty < x < \infty$.

(a) Find the value of the constant c.

(b) Find the probability that X^2 lies between 1/3 and 1.

(c) Find the distribution function.

Solution:

(a) We must have $\int_{-\infty}^{\infty} f(x)dx = 1$, i.e,

$$\int_{-\infty}^{\infty} c/(x^2 + 1)dx = c\tan^{-1} x|_{-\infty}^{\infty} = c[\pi/2 - (-\pi/2)] = 1$$

so that $c = (1/\pi)$.

(b) If $1/3 \le X^2 \le 1$, then either $\sqrt{3}/3 \le X \le 1$ or $-1 \le X \le -\sqrt{3}/3$. Thus the required probability is

$$(1/\pi)\int_{-1}^{-\sqrt{3}/3} \frac{dx}{x^2 + 1} + (1/\pi)\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2 + 1} =$$

$$= (2/\pi)\int_{\sqrt{3}/3}^{1} \frac{dx}{x^2 + 1} = (2/\pi)[\tan^{-1}(1) - \tan^{-1}(\sqrt{3}/3)] = (2/\pi)(\pi/4 - \pi/6) = 1/6$$

(c)

$$F(x) = \int_{-\infty}^{x} f(u)du = (1/\pi)\int_{-\infty}^{x} \frac{du}{u^2 + 1} = (1/\pi)[\tan^{-1}u\Big|_{-\infty}^{x} =$$

$$= (1/\pi)[tan^{-1}x - tan^{-1}(-\infty)] = (1/\pi)[tan^{-1}x + \pi/2] = 1/2 + (1/\pi)tan^{-1}x$$

Problem 2

The mean weight of 500 male students at a certain college is 151lb and the standard deviation is 15 lb. Assuming that the weights are normally distributed, find how many students weigh (a) between 120 and 155 lb, (b) more than 185 lb.

Solution:

If W denotes the weight of a student chosen at random, then W has Gaussian distribution with $(\mu = 151, \sigma^2 = 15^2)$.

The CDF of W is:

$$F_W(w) = Pr(W \leq w) = \Phi\left(\frac{w - 151}{15}\right)$$

(a)

$$\begin{aligned} Pr(120 \leq W \leq 155) &= F_W(155) - F_W(120) \\ &= \Phi\left(\frac{155 - 151}{15}\right) - \Phi\left(\frac{120 - 151}{15}\right) \\ &= \Phi(0.266) - \Phi(-2.067) \\ &= \Phi(0.266) - (1 - \Phi(2.067)) \\ &\approx 0.6026 - (1 - 0.98077) \approx 0.584 \end{aligned}$$

The number of students weighing between 120 and 155 lb is approximately $500(0.584) = 292$.

(b)

$$\begin{aligned} Pr(W > 185) &= 1 - Pr(W \leq 185) \\ &= 1 - F_W(185) \\ &= 1 - \Phi\left(\frac{185 - 151}{15}\right) \\ &= 1 - \Phi(2.267) \\ &\approx 1 - 0.9884 \approx 0.012 \end{aligned}$$

The number of students weighing more than 185 lb is approximately $500(0.012) = 6$.

Problem 3

In the mid to late 1980's, in response to the growing AIDS crisis and the emergence of new, highly sensitive tests for the virus, there were a number of calls for widespread public screening for the disease. The focus at the time was the sensitivity and specificity (roughly, 1-false positive rate) of the tests at hand. For the tests in question the sensitivity was Pr(Positive Test | Infected) ≈ 1 and the false positive rate was Pr(Positive Test | Uninfected) $\approx .00005$ - an unusually low false positive rate. What was generally neglected in the debate, however, was the low prevelence of the disease in the general population: Pr(Infected) ≈ 0.0001. Since being told you are HIV positive has dramatic ramifications, what clearly matters to you as an individual is the probability that you are uninfected given a positive test result: Pr(Uninfected | Positive test). Calculate this probability. Would you volunteer for such screening? How does this number change if you are in a "high risk" population - i.e. if Pr(Infected) is significantly higher?

Solution:

$$\Pr(\text{Uninfected} \mid \text{Positive test}) = \frac{\Pr(\text{Positive test} \mid \text{Uninfected})\Pr(\text{Uninfected})}{\Pr(\text{Positive test})} =$$

$$= \frac{\Pr(\text{Positive test} \mid \text{Uninfected})\Pr(\text{Uninfected})}{\Pr(\text{Positive test} \mid \text{Uninfected})\Pr(\text{Uninfected}) + \Pr(\text{Positive test} \mid \text{Infected})\Pr(\text{Infected})} =$$

$$= \frac{0.00005(1 - .0001)}{0.00005(1 - .0001) + 1(.0001)} = 0.3333$$

So there is a 1/3 probability that you are actually healthy if you are in a low risk population and your test is positive!

If Pr(Infected) is significantly higher, the probability you are actually healthy given a positive test result rapidly decreases to essentially zero. For example if Pr(Infected) = 0.001 (.1% prior probability of infection), Pr(Uninfected | Positive test) = 5% while if Pr(Infected) = 0.01 (1% prior probability of infection), Pr(Uninfected | Positive test) = .5%.