Recitation 6

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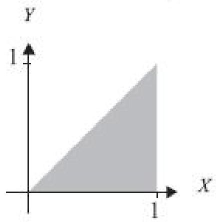
Problem 1: [YG] Problem 4.6.9

Solution:

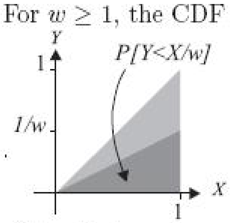
Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Since $f_{X,Y}(x,y) = 0$ for $y > x$, we can conclude that $Y \le X$ and that $W = X/Y \ge 1$. Since Y can be arbitrarily small but positive, W can be arbitrarily large. Hence the range of W is $S_W = \{w | w \ge 1\}$.

(b) For $w \ge 1$, the CDF of W is



$$\begin{aligned} F_W(w) &= P[X/Y \le w] & (2)\\ &= 1 - P[X/Y > w] & (3)\\ &= 1 - P[Y < X/w] & (4)\\ &= 1 - 1/w & (5) \end{aligned}$$

Note that we have used the fact that $P[Y < X/w]$ equals 1/2 times the area of the corresponding triangle. The complete CDF is

$$F_W(w) = \begin{cases} 0 & w < 1 \\ 1 - 1/w & w \ge 1 \end{cases} \tag{6}$$

The PDF of W is found by differentiating the CDF.

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 1/w^2 & w \ge 1 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

To find the expected value $E[W]$, we write

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = \int_{1}^{\infty} \frac{dw}{w}. \tag{8}$$

However, the integral diverges and $E[W]$ is undefined.

Problem 2: [YG] Problem 4.7.8

Solution:

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} (x+y)/3 & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Before calculating moments, we first find the marginal PDFs of X and Y. For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^2 \frac{x+y}{3}\, dy = \frac{xy}{3} + \frac{y^2}{6}\Big|_{y=0}^{y=2} = \frac{2x+2}{3} \tag{2}$$

For $0 \le y \le 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_0^1 \left(\frac{x}{3} + \frac{y}{3}\right) dx = \frac{x^2}{6} + \frac{xy}{3}\Big|_{x=0}^{x=1} = \frac{2y+1}{6} \tag{3}$$

Complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} \frac{2x+2}{3} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{2y+1}{6} & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(a) The expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x\frac{2x+2}{3}\, dx = \frac{2x^3}{9} + \frac{x^2}{3}\Big|_0^1 = \frac{5}{9} \tag{5}$$

The second moment of X is

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2\frac{2x+2}{3}\, dx = \frac{x^4}{6} + \frac{2x^3}{9}\Big|_0^1 = \frac{7}{18} \tag{6}$$

The variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 7/18 - (5/9)^2 = 13/162$.

(b) The expected value of Y is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^2 y\frac{2y+1}{6}\, dy = \frac{y^2}{12} + \frac{y^3}{9}\Big|_0^2 = \frac{11}{9} \tag{7}$$

The second moment of Y is

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^2 y^2\frac{2y+1}{6}\, dy = \frac{y^3}{18} + \frac{y^4}{12}\Big|_0^2 = \frac{16}{9} \tag{8}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 23/81$.

(c) The correlation of X and Y is

$$E[XY] = \iint xy f_{X,Y}(x,y)\, dx\, dy \tag{9}$$

$$= \int_0^1 \int_0^2 xy\left(\frac{x+y}{3}\right) dy\, dx \tag{10}$$

$$= \int_0^1 \left(\frac{x^2y^2}{6} + \frac{xy^3}{9}\Big|_{y=0}^{y=2}\right) dx \tag{11}$$

$$= \int_0^1 \left(\frac{2x^2}{3} + \frac{8x}{9}\right) dx = \frac{2x^3}{9} + \frac{4x^2}{9}\Big|_0^1 = \frac{2}{3} \tag{12}$$

The covariance is $\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = -1/81$.

(d) The expected value of X and Y is

$$E[X+Y] = E[X] + E[Y] = 5/9 + 11/9 = 16/9 \tag{13}$$

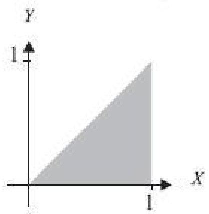
(e) By Theorem 4.15,

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = \frac{13}{162} + \frac{23}{81} - \frac{2}{81} = \frac{55}{162} \tag{14}$$

Problem 3: [YG] Problem 4.9.5

Solution:

Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For $0 \le x \le 1$, the marginal PDF for X satisfies

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^x 2\, dy = 2x \tag{2}$$

Note that $f_X(x) = 0$ for $x < 0$ or $x > 1$. Hence the complete expression for the marginal PDF of X is

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional PDF of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} 1/x & 0 \le y \le x \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Given $X = x$, Y has a uniform PDF over $[0, x]$ and thus has conditional expected value $E[Y|X = x] = x/2$. Another way to obtain this result is to calculate $\int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy$.