Recitation 7

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Problem 1: [YG] Problem 5.3.4

Solution:

For $0 \le y_1 \le y_4 \le 1$, the marginal PDF of Y_1 and Y_4 satisfies

$$f_{Y_1,Y_4}(y_1,y_4) = \iint f_{\mathbf{Y}}(\mathbf{y})\ dy_2\,dy_3 \tag{1}$$

$$= \int_{y_1}^{y_4}\left(\int_{y_2}^{y_4} 24\,dy_3\right)\ dy_2 \tag{2}$$

$$= \int_{y_1}^{y_4} 24(y_4 - y_2)\,dy_2 \tag{3}$$

$$= -12(y_4-y_2)^2\Big|_{y_2=y_1}^{y_2=y_4} = 12(y_4-y_1)^2 \tag{4}$$

The complete expression for the joint PDF of Y_1 and Y_4 is

$$f_{Y_1,Y_4}(y_1,y_4) = \begin{cases} 12(y_4-y_1)^2 & 0 \le y_1 \le y_4 \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

For $0 \le y_1 \le y_2 \le 1$, the marginal PDF of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1,y_2) = \iint f_{\mathbf{Y}}(\mathbf{y})\ dy_3\,dy_4 \tag{6}$$

$$= \int_{y_2}^{1}\left(\int_{y_3}^{1} 24\,dy_4\right)\ dy_3 \tag{7}$$

$$= \int_{y_2}^{1} 24(1-y_3)\,dy_3 = 12(1-y_2)^2 \tag{8}$$

The complete expression for the joint PDF of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} 12(1-y_2)^2 & 0 \le y_1 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

For $0 \le y_1 \le 1$, the marginal PDF of Y_1 can be found from

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\ dy_2 = \int_{y_1}^{1} 12(1-y_2)^2\,dy_2 = 4(1-y_1)^3 \tag{10}$$

The complete expression of the PDF of Y_1 is

$$f_{Y_1}(y_1) = \begin{cases} 4(1-y_1)^3 & 0 \le y_1 \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Note that the integral $f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{Y_1,Y_4}(y_1,y_4)\,dy_4$ would have yielded the same result. This is a good way to check our derivations of $f_{Y_1,Y_4}(y_1,y_4)$ and $f_{Y_1,Y_2}(y_1,y_2)$.

Problem 2: [YG] Problem 5.5.4

Solution:

Let X_i denote the finishing time of boat i. Since finishing times of all boats are iid Gaussian random variables with expected value 35 minutes and standard deviation 5 minutes, we know that each X_i has CDF

$$F_{X_i}(x) = P[X_i \leq x] = P\left[\frac{X_i - 35}{5} \leq \frac{x - 35}{5}\right] = \Phi\left(\frac{x-35}{5}\right) \tag{1}$$

(a) The time of the winning boat is

$$W = \min(X_1, X_2, \ldots, X_{10}) \tag{2}$$

To find the probability that $W \leq 25$, we will find the CDF $F_W(w)$ since this will also be useful for part (c).

$$\begin{aligned} F_W(w) &= P[\min(X_1, X_2, \ldots, X_{10}) \leq w] & (3)\\ &= 1 - P[\min(X_1, X_2, \ldots, X_{10}) > w] & (4)\\ &= 1 - P[X_1 > w, X_2 > w, \ldots, X_{10} > w] & (5) \end{aligned}$$

Since the X_i are iid,

$$\begin{aligned} F_W(w) = 1 - \prod_{i=1}^{10} P[X_i > w] &= 1 - (1 - F_{X_i}(w))^{10} & (6)\\ &= 1 - \left(1 - \Phi\left(\frac{w-35}{5}\right)\right)^{10} & (7) \end{aligned}$$

Thus,

$$\begin{aligned} P[W \leq 25] = F_W(25) &= 1 - (1 - \Phi(-2))^{10} & (8)\\ &= 1 - [\Phi(2)]^{10} = 0.2056. & (9) \end{aligned}$$

(b) The finishing time of the last boat is $L = \max(X_1, \ldots, X_{10})$. The probability that the last boat finishes in more than 50 minutes is

$$P[L > 50] = 1 - P[L \leq 50] \tag{10}$$
$$= 1 - P[X_1 \leq 50, X_2 \leq 50, \ldots, X_{10} \leq 50] \tag{11}$$

Once again, since the X_i are iid Gaussian $(35, 5)$ random variables,

$$P[L > 50] = 1 - \prod_{i=1}^{10} P[X_i \leq 50] = 1 - (F_{X_i}(50))^{10} \tag{12}$$
$$= 1 - (\Phi([50 - 35]/5))^{10} \tag{13}$$
$$= 1 - (\Phi(3))^{10} = 0.0134 \tag{14}$$

(c) A boat will finish in negative time if and only iff the winning boat finishes in negative time, which has probability

$$F_W(0) = 1 - (1 - \Phi(-35/5))^{10} = 1 - (1 - \Phi(-7))^{10} = 1 - (\Phi(7))^{10}. \tag{15}$$

Unfortunately, the tables in the text have neither $\Phi(7)$ nor $Q(7)$. However, those with access to MATLAB, or a programmable calculator, can find out that $Q(7) = 1 - \Phi(7) = 1.28 \times 10^{-12}$. This implies that a boat finishes in negative time with probability

$$F_W(0) = 1 - (1 - 1.28 \times 10^{-12})^{10} = 1.28 \times 10^{-11}. \tag{16}$$