Recitation 8

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Problem 1: Functions of two RVs

$$W = g(X, Y) \qquad F_W(w) = P[W \le w] = \int\int_{g(x,y)\le w} f_{X,Y}(x, y)dxdy$$

$W = max(X, Y) \Rightarrow F_W(w) = P[W \le w] = P[X \le w, Y \le w] = \int_{-\infty}^{w}\int_{-\infty}^{w} f_{X,Y}(x, y)dxdy$
$f_W(w) = \frac{dF_W(w)}{dw}$

$L = min(X, Y) \Rightarrow F_L(l) = P[L \le l] = 1 - P[L > l] = 1 - P[X > l, Y > l] = 1 - \int_{l}^{\infty}\int_{l}^{\infty} f_{X,Y}(x, y)dxdy$
$f_L(l) = \frac{dF_L(l)}{dl}$

X and Y independent, $U = X + Y \Rightarrow f_U(u) = \int_{-\infty}^{\infty} f_X(v)f_Y(u - v)dv$
Example: X and Y independent, $U = X + Y$,

$$f_X(v) = f_Y(v) = \begin{cases} e^{-v} & v \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_U(u) = \int_{-\infty}^{\infty} f_X(v)f_Y(u - v)dv = \begin{cases} ue^{-u} & u \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 2: [YG] Problems 5.7.1, 5.7.2

(a) From Theorem 5.12, the correlation matrix of $\mathbf{X}$ is

$$\mathbf{R}_X = \mathbf{C}_X + \boldsymbol{\mu}_X\boldsymbol{\mu}_X' \tag{1}$$

$$= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}\begin{bmatrix} 4 & 8 & 6 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 16 & 32 & 24 \\ 32 & 64 & 48 \\ 24 & 48 & 36 \end{bmatrix} = \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix} \tag{3}$$

(b) Let $\mathbf{Y} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$. Since $\mathbf{Y}$ is a subset of the components of $\mathbf{X}$, it is a Gaussian random vector with expected velue vector

$$\boldsymbol{\mu}_Y = \begin{bmatrix} E[X_1] & E[X_2] \end{bmatrix}' = \begin{bmatrix} 4 & 8 \end{bmatrix}'. \tag{4}$$

and covariance matrix

$$\mathbf{C}_Y = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] \\ \mathbf{C}_{X_1 X_2} & \mathrm{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \tag{5}$$

We note that $\det(\mathbf{C}_Y) = 12$ and that

$$\mathbf{C}_Y^{-1} = \frac{1}{12} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}. \tag{6}$$

This implies that

$$(\mathbf{y} - \boldsymbol{\mu}_Y)' \mathbf{C}_Y^{-1} (\mathbf{y} - \boldsymbol{\mu}_Y) = \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} y_1 - 4 \\ y_2 - 8 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} y_1/3 + y_2/6 - 8/3 \\ y_1/6 + y_2/3 - 10/3 \end{bmatrix} \tag{8}$$

$$= \frac{y_1^2}{3} + \frac{y_1 y_2}{3} - \frac{16 y_1}{3} - \frac{20 y_2}{3} + \frac{y_2^2}{3} + \frac{112}{3} \tag{9}$$

The PDF of $\mathbf{Y}$ is

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2\pi\sqrt{12}} e^{-(\mathbf{y}-\boldsymbol{\mu}_Y)'\mathbf{C}_Y^{-1}(\mathbf{y}-\boldsymbol{\mu}_Y)/2} \tag{10}$$

$$= \frac{1}{\sqrt{48\pi^2}} e^{-(y_1^2 + y_1 y_2 - 16 y_1 - 20 y_2 + y_2^2 + 112)/6} \tag{11}$$

Since $\mathbf{Y} = \begin{bmatrix} X_1, X_2 \end{bmatrix}'$, the PDF of X_1 and X_2 is simply

$$f_{X_1, X_2}(x_1, x_2) = f_{Y_1, Y_2}(x_1, x_2) = \frac{1}{\sqrt{48\pi^2}} e^{-(x_1^2 + x_1 x_2 - 16 x_1 - 20 x_2 + x_2^2 + 112)/6} \tag{12}$$

(c) We can observe directly from $\boldsymbol{\mu}_X$ and $\mathbf{C}_X$ that X_1 is a Gaussian $(4, 2)$ random variable. Thus,

$$P[X_1 > 8] = P\left[\frac{X_1 - 4}{2} > \frac{8 - 4}{2}\right] = Q(2) = 0.0228 \tag{13}$$

We are given that $\mathbf{X}$ is a Gaussian random vector with

$$\boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} \qquad \mathbf{C}_{\mathbf{X}} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}. \tag{1}$$

We are also given that $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}. \tag{2}$$

Since the two rows of $\mathbf{A}$ are linearly independent row vectors, $\mathbf{A}$ has rank 2. By Theorem 5.16, $\mathbf{Y}$ is a Gaussian random vector. Given these facts, the various parts of this problem are just straightforward calculations using Theorem 5.16.

(a) The expected value of $\mathbf{Y}$ is

$$\boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}} + \mathbf{b} = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} + \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix}. \tag{3}$$

(b) The covariance matrix of $\mathbf{Y}$ is

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{A}\mathbf{C}_{\mathbf{X}}\mathbf{A} \tag{4}$$

$$= \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1/2 & -1/2 \\ 2/3 & 2/3 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 43 & 55 \\ 55 & 103 \end{bmatrix}. \tag{5}$$

(c) $\mathbf{Y}$ has correlation matrix

$$\mathbf{R}_{\mathbf{Y}} = \mathbf{C}_{\mathbf{Y}} + \boldsymbol{\mu}_{\mathbf{Y}}\boldsymbol{\mu}'_{\mathbf{Y}} = \frac{1}{9} \begin{bmatrix} 43 & 55 \\ 55 & 103 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \end{bmatrix} \begin{bmatrix} 8 & 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 619 & 55 \\ 55 & 103 \end{bmatrix} \tag{6}$$

(d) From $\boldsymbol{\mu}_{\mathbf{Y}}$, we see that $E[Y_2] = 0$. From the covariance matrix $\mathbf{C}_{\mathbf{Y}}$, we learn that Y_2 has variance $\sigma_2^2 = C_{\mathbf{Y}}(2,2) = 103/9$. Since Y_2 is a Gaussian random variable,

$$P[-1 \leq Y_2 \leq 1] = P\left[-\frac{1}{\sigma_2} \leq \frac{Y_2}{\sigma_2} \leq \frac{1}{\sigma_2}\right] \tag{7}$$

$$= \Phi\left(\frac{1}{\sigma_2}\right) - \Phi\left(\frac{-1}{\sigma_2}\right) \tag{8}$$

$$= 2\Phi\left(\frac{1}{\sigma_2}\right) - 1 \tag{9}$$

$$= 2\Phi\left(\frac{3}{\sqrt{103}}\right) - 1 = 0.2325. \tag{10}$$

Problem 3: (from EC505)

In a radar system, a decision about the presence or absence of a target is made on the basis of an observation Y. If the target is present, $Y = A + X$, where $A > 0$ is a known constant. If the target is not present, $Y = X$. X has normal distribution: $X \sim N(0, N_0)$. The *a priori* probability that the target is not present is 0.999.

(a) Derive the MAP decision rule for detecting the target and the associated probability of error.

(b) Now, assume that missing the target is ten times worse than falsely detecting a target: $C_{00} = C_{11} = 0$, $C_{01} = 10$ and $C_{10} = 1$, where H_0 is "target absent" and H_1 is "target present". What is the decision rule which minimizes the conditional risk for y?

(c) Sketch a hypothetical ROC curve for this problem, illustrating the two points corresponding to the two decision rules. Make a sketch to illustrate qualitatively how the ROC curve would change if you were to have 5 independent observations to base your decision on. Make a sketch to illustrate qualitatively how the ROC curve would change if the noise level increased (i.e. A/N_0 decreased).

Solution:

(a) Note that given the statements in the problem we have:

$$\begin{aligned} P(y \mid H_0) &= N(y; 0, N_0) \\ P(y \mid H_1) &= N(y; A, N_0) \end{aligned}$$

and $P_0 = 0.999$, $P_1 = 0.001$. Now the MAP rule is given by:

$$P(H_1 \mid y) \underset{H_0}{\overset{H_1}{\gtrless}} P(H_0 \mid y)$$

$$\begin{aligned} \Longrightarrow \quad P(y \mid H_1)P_1 &\underset{H_0}{\overset{H_1}{\gtrless}} P(y \mid H_0)P_0 \\ \Longrightarrow \quad y &\underset{H_0}{\overset{H_1}{\gtrless}} \frac{2N_0 \ln\left(\frac{P_0}{P_1}\right) + A^2}{2A} = \Gamma_{\text{MAP}} \end{aligned}$$

where we have used the results for Gaussian problems given in class to simplify the last expression. Now the corresponding probability of error is given by:

$$\begin{aligned} \Pr(\varepsilon) &= P_F P_0 + P_M P_1 = \Pr(y > \Gamma_{\text{MAP}} \mid H_0)P_0 + \Pr(y < \Gamma_{\text{MAP}} \mid H_1)P_1 \\ &= P_0 \int_{\Gamma_{\text{MAP}}}^{\infty} P(y \mid H_0)\, dy + P_1 \int_{-\infty}^{\Gamma_{\text{MAP}}} P(y \mid H_1)\, dy \\ &= P_0 \int_{\Gamma_{\text{MAP}}}^{\infty} N(y; 0, N_0)\, dy + P_1 \int_{-\infty}^{\Gamma_{\text{MAP}}} N(y; A, N_0)\, dy \\ &= P_0 Q\left(\frac{\Gamma_{\text{MAP}}}{\sqrt{N_0}}\right) + P_1 Q\left(-\frac{\Gamma_{\text{MAP}} - A}{\sqrt{N_0}}\right) \end{aligned}$$

(b) In this case we have for the decision rule:

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{2N_0 \ln\left(\frac{P_0}{10P_1}\right) + A^2}{2A}$$

Note that this change to the cost structure serves to <u>increase P_D</u>. This makes sense, since we have raised the cost for missing the target, we would expect that the detection probability should rise.

(c) The ROC for the various cases is illustrated below. Note the following:

- The rule in part (b) will have a higher P_D than the rule in part (a), but they will be on the same ROC, since the rules differ only in their choice of threshold.
- If we have more measurements, P_D must be higher at any P_F than for the ROC obtained in parts (a),(b) – i.e. the ROC for this case will be above the ROC obtained for these parts.
- If the noise is increased, P_D must be lower at any P_F than the ROC obtained in parts (a),(b) – i.e. the ROC for this case will be below the ROC obtained for these parts.