Recitation 9

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Problem 1: [YG] Problem 9.1.3 a-c

Solution:

(a) For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_x^1 2\, dy = 2(1-x). \tag{1}$$

The complete expression of the PDF of X is

$$f_X(x) = \begin{cases} 2(1-x) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) The blind estimate of X is

$$\hat{X}_B = E[X] = \int_0^1 2x(1-x)\, dx = \left(x^2 - \frac{2x^3}{3} \right)\Big|_0^1 = \frac{1}{3}. \tag{3}$$

(c) First we calculate

$$P[X > 1/2] = \int_{1/2}^1 f_X(x)\, dx = \int_{1/2}^1 2(1-x)\, dx = (2x - x^2)\big|_{1/2}^1 = \frac{1}{4}. \tag{4}$$

Now we calculate the conditional PDF of X given $X > 1/2$.

$$f_{X|X>1/2}(x) = \begin{cases} \frac{f_X(x)}{P[X>1/2]} & x > 1/2, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 8(1-x) & 1/2 < x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The minimum mean square error estimate of X given $X > 1/2$ is

$$E[X|X > 1/2] = \int_{-\infty}^{\infty} x f_{X|X>1/2}(x)\, dx \tag{6}$$

$$= \int_{1/2}^1 8x(1-x)\, dx = \left(4x^2 - \frac{8x^3}{3} \right)\Big|_{1/2}^1 = \frac{2}{3}. \tag{7}$$

Problem 2: [YG] Problem 6.6.1

Solution:

We know that the waiting time, W is uniformly distributed on [0,10] and therefore has the following PDF.

$$f_W(w) = \begin{cases} 1/10 & 0 \leq w \leq 10 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

We also know that the total time is 3 milliseconds plus the waiting time, that is $X = W + 3$.

(a) The expected value of X is $E[X] = E[W+3] = E[W] + 3 = 5 + 3 = 8$.

(b) The variance of X is $\text{Var}[X] = \text{Var}[W+3] = \text{Var}[W] = 25/3$.

(c) The expected value of A is $E[A] = 12E[X] = 96$.

(d) The standard deviation of A is $\sigma_A = \sqrt{\text{Var}[A]} = \sqrt{12(25/3)} = 10$.

(e) $P[A > 116] = 1 - \Phi(\frac{116-96}{10}) = 1 - \Phi(2) = 0.02275$.

(f) $P[A < 86] = \Phi(\frac{86-96}{10}) = \Phi(-1) = 1 - \Phi(1) = 0.1587$

Problem 2

a X_i takes the value 1 with probability p and the value 0 with probability $1-p$. Let $S_n = \sum_{i=1}^{n} X_i$ where the Xi's are mutually independent and identically distributed. Compute (i) the probability mass-function (pmf), (ii) the mean, and (iii) the variance of S_n.

b Suppose X_i's are binary digits with probabilities $p(1) = 0.005$ and $p(0) = 0.995$. The digits are taken 100 at a time and a codeword is provided for every sequence of 100 digits containing three or fewer 1's. (i) Calculate the probability of observing a sequence for which no codeword has been assigned. (ii) Use Chebyshev's inequality to bound the probability of observing a sequence for which no codeword has been assigned. Compare the bound with the actual probability computed in (i).

Solution:

a (i) The sum of independent Bernoulli random variables is a Binomial random variable.

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

(ii)

$$E[S_n] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = nE[X_1] = np$$

(iii)

$$Var[S_n] = Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i] = nVar[X_1] = np(1-p)$$

b (i) The probability that a 100-bit sequence has three or fewer ones is

$$\sum_{i=0}^{3} \binom{100}{i} (0.005)^i (0.995)^{100-i} = 0.99833$$

Thus the probability that the sequence that is generated cannot be encoded is $1 - 0.99833 = 0.00167$.

(ii) In the case of a random variable S_n that is the sum of n i.i.d random variables $X_1, ..., X_n$, Chebyshev's inequality states that

$$P(|S_n - n\mu| \geq a) \leq \frac{n\sigma^2}{a^2}$$

where μ and σ^2 are the mean and variance of the X's (therefore $n\mu$ and $n\sigma^2$ are the mean and variance of S_n). In this problem, $n = 100$, $\mu = p = 0.005$ and $\sigma^2 = p(1-p) = (0.005)(0.995)$. Further note that $S_{100} \geq 4$ if and only if $|S_{100} - 100(0.005)| \geq 3.5$, so we should choose $a = 3.5$. Then,

$$P(S_{100} \geq 4) \leq \frac{100(0.005)(0.995)}{(3.5)^2} \approx 0.04061$$

This bound is about 24 times bigger than the actual probability 0.00167. This shows that the Chebyshev inequality does not truly reflect the rate of decay of probability with blocklength n.