EC381/MN308
Probability and Some Statistics

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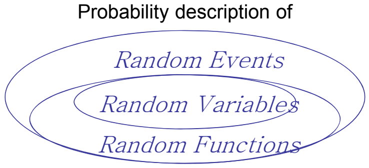
Lecture 1 - Outline

1. Administrative stuff.
2. Scope of the course and Overview.

SCOPE OF THE COURSE

Probability description of



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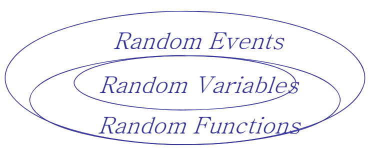
Inference, predictions, and decisions based on measurements of:

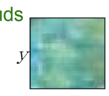


Examples of Random Events:
- The outcome of a tossed coin is a head
- The waiting time at a toll booth is greater than 5 minutes
- An electronic device fails after 1000 hours
- The production lead time for a product is less than 1 week
- An energy level $E = 1.5\ E_f$ in a semiconductor is occupied

Examples of Random Variables:
- The number of Geiger-counter counts N in 1 sec (N = 0,1,2,..)
- The waiting time τ to access a network ($0 < \tau < \infty$)
- The lifetime (time to failure) τ of a hard-disk ($0 < \tau < \infty$)
- The customer demand D for a product (units/week, D=0,1,…)
- The resistance R of a resistor picked from a box of resistors rated at 1 kΩ ± 5% ($0 < R < \infty$)

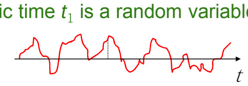
Examples of Random Functions:
- Temperature as a function of time $T(t)$
- Voltage $V(t)$ measured across a noisy amplifier as a function of time

- An image $I(x,y)$ of clouds



The value of a *random function* at a specific value of its argument is a *random variable*.

For example, $V(t_1)$ at a specific time t_1 is a random variable (we may call it V_1)



A *random function* is therefore composed of a large number of *random variables*.

Random variables may be described in terms of *random events* associated with them.

Example:
The event that a random variable X has a value smaller than or equal to some value x.

Similarly, *random functions* may be described in terms of *random events* associated them.

Example:
The event that a random function $X(t)$ has a value smaller than x_1 at t_1 and a value smaller than x_2 at t_2.

Probability description of

Probability theory is a “calculus” for representing and for reasoning about the uncertainty of random events, random variables, and random functions

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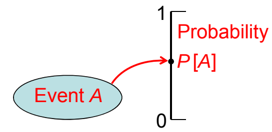
A *random event* A is characterized by a number between 0 and 1, called the probability of the event $P[A]$.

If $P[A] = 0$, the event never occurs (is impossible)
If $P[A] = 1$, the event always occurs (is certain or sure)

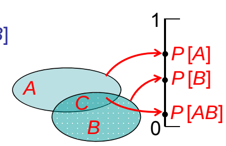
Otherwise, the event occurs sometimes
The larger $P[A]$ is, the more likely is the event to occur



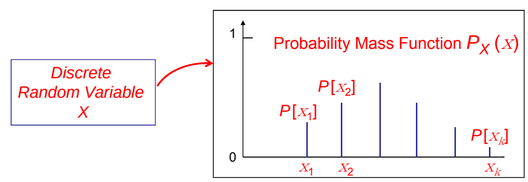
Two random events A *and* B are characterized by the probabilities of the individual events, $P[A]$ and $P[B]$, and also by the probability $P[C]$ of the joint event $C = AB$ (i.e., A and B).

Independence
The event A and the event B are said to be independent if, and only if, the joint probability is the product of the individual probabilities:

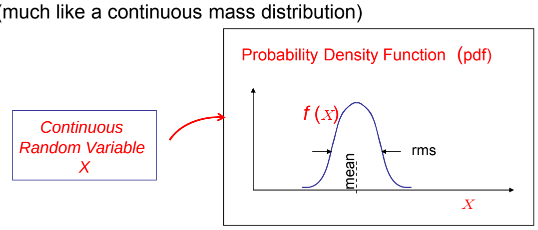
$$P[AB] = P[A]\, P[B]$$



A *random variable* X that takes one of k possible values $\{x_1, x_2, \cdots, x_k\}$ is characterized by the probabilities $\{P[x_1], P[x_2], \ldots, P[x_k]\}$ of the events $\{X = x_1, X = x_2, \ldots, X = x_k\}$.



A *random variable* X that takes continuous values $\{x\}$ is characterized by a continuous probability distribution (much like a continuous mass distribution)



The centroid of the distribution is called the **mean**.
The width of the distribution, called the **rms** value, is a measure of the degree of uncertainty of the random variable

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A *random function* $X(t)$ is made of a set of random variables at $\{t_1, t_2, \ldots, t_k\}$, each characterized by a probability description.. More later!

HOW TO DETERMINE THE PROBABILITY OF AN EVENT

Method 1: Incidence

Observe the random event a large number of times N.
Count the number of times the event occurs N_A.
The ratio N_A/N, called the relative frequency, is a measure of P[A]. This is the statistician's empirical approach.

The mathematician's approach: Take the limit $N \to \infty$

$$P[A] = \lim_{N \to \infty} \frac{N_A}{N}$$

How does the sample size of a poll affect the reliability?

Sometimes, the experiment cannot be done (e.g., the probability of failure in a nuclear facility)

Method 2: Subjective belief based on some reasonable principle

Principle of "equality" or "nonprejudice"
If there is no reason to believe that some events are more probable than others, assume they are equally probable

Principle of maximum entropy (ignorance) in physics

Method 3: Computations: random events caused by other underlying random events

If the occurrence of an event C is triggered by two other events A and B, or A or B, the probability of C may be computed using a calculus of probability.

Example: the overall reliability of a system is derived from the reliability of its components.

ORIGIN OF RANDOMNESS (UNCERTAINTY)

- Phenomena whose outcomes are too hard to model because they involve too many factors -- or factors that are difficult to model physically
  - Probability of a coin coming up with a head or a tail: physics too detailed with too few observables.
  - Likelihood that the Patriots will win the Super Bowl: not clear how to analyze the problem because too complex; more belief than physics.
  - Variability of resistor value in a batch of products.
  - Measurement errors/sampling.
- Phenomena that are inherently random (quantum)

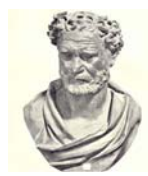
"God does not play dice [with the universe]."

Einstein

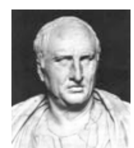
- A random model may be used to represent a state of ignorance

PROBABILITY PERSPECTIVES

Ask our elders



Democritus



Cicero

- *Everything existing in the universe is the fruit of chance.*
- *Probability is the very guide of life.*

Ask a mathematician

- *It is remarkable that a science that began with the consideration of games of chance should have become the most important object of human knowledge.*
- *The most important questions of life are indeed, for the most part, really only problems of probability.*
- *Probability theory is nothing but common sense reduced to calculation.*

Pierre Simon Marquis de Laplace

Théorie Analytique des Probabilités, 1812

Ask scientists

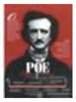
James Clerk Maxwell

Max Born

- *.. the true logic for this world is the Calculus of Probabilities.*
- *The conception of chance enters in the very first steps of scientific activity by virtue of the fact that no observation is absolutely correct.*

Ask a poet/writer

Coincidences, in general, are great stumbling blocks in the way of that class of thinkers who have been educated to know nothing of the theory of probabilities -- that theory to which the most glorious objects of human research are indebted for the most glorious of illustrations.



Edgar Allan Poe, "The Murders in the Rue Morgue"

An MD/writer's View

I cannot make my peace with the randomness doctrine: I cannot abide the notion of purposelessness and blind chance in nature. And yet I do not know what to put in its place for the quieting of my mind.

Lewis Thomas, On the Uncertainty of Science

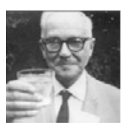
In the view of statisticians and probability theorists

.. the vast majority of scientists and even non-scientists, are convinced that they know what "random" is .. If pressed for a definition of randomness, most people will fall back on such formulation as "lack of regularity" or "unpredictability." But they are then faced with the equally difficult task of defining "regularity" or "predictability," and soon find themselves immersed in metaphysics.

Mark Kac, "What is random," *American Scientist*, vol. 71, July-August, 1983

Each morning, before breakfast, every single one of us approaches an urn filled with white and black balls. We draw a ball. If it is white, we survive the day. If it is black, we die. The proportion of black balls in the urn is not the same for each day, but grows as we become older. Still, there are always some white balls present and some of us continue to draw them day after day for many years.

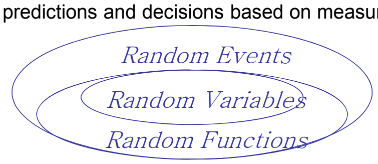
J. Neyman and E. L. Scott, "The distribution of galaxies."

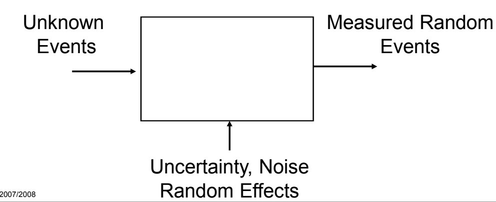


Jerzy Neyman, one of the most famous statisticians and probability theorist of this century, died on August 5, 1981, at the age of 87.

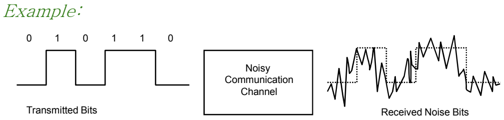
INFERENCE

Inference, predictions and decisions based on measurements of:





Example:

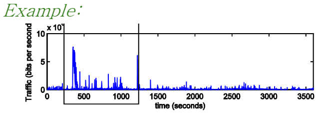


Inference: Decide whether the transmitted bit is 0 or 1 based on the random Measurement of the received bit. What are the errors involved? This is a hypothesis testing problem.

Example:

Decide whether this cellular image is cancerous or not. What are the errors involved?

Example:



Decide whether this Internet trace contains a DoS attack.

Is Probability Intuitive?

- **Birthday Paradox:** What is the probability that two or more among us have the same birthday?
- **The Monty Hall Puzzle:** You are a contestant in a show and offered a choice between 3 doors. Open the correct one, and you win a car; open either of the others and you win a goat. You pick one door. The host now opens one of the other two doors to reveal a goat. You are offered the opportunity to switch to the remaining door. Do you stick with your original choice or decide to switch?
- **The false positive puzzle:**
  - In random testing you test positive for a disease.
  - The test has 5% false alarm probability and 0% misdetection probability.
  - In the population at large, 1 in 1000 has the disease.

Should you be concerned? What is the probability of having the disease?

Some History

• A gambler's dispute in 1654 led to the creation of Probability Theory by Pascal and Fermat. Huygens (1657): first book on Probability based on work of Pascal and Fermat.

• Significant developments in the18th century by Jakob Bernoulli and de Moivre.

• Laplace (1812) significantly generalized the theory, defined probability for experiments with finite equally likely outcomes.

• Poisson (1837) presents his distribution.

• Chebyshev mid 19th century proves his inequality.

• Fisher and von Mises (1920's): statistical or empirical probability defined by the frequency of an outcome.

• de Finetti (1936): "subjective probability".

• Kolmogorov (1933): "axiomatic probability".

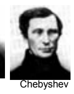

Pascal


Fermat


Bernoulli


de Moivre


Poisson


Chebyshev


Kolmogorov

Some Applications

• Communications and Networks (design, performance analysis, traffic modeling, coding, physical layer, social/biological networks)

• Manufacturing Systems and Supply Chains (demand/production modeling, performance analysis, quality control, product development, design of experiments)

• Finance (asset price modeling, pricing derivatives, risk management)

• Medicine (effectiveness of drugs, treatments, procedures; evaluate impact of lifestyle, environmental factors; assess risks)

• Insurance (set premiums, risk control)

• Marketing and Retail management (surveys, polling, revenue management, pricing, modeling consumer choice)

• Strategic decision making (game theory, stochastic control)