# EC381/MN308
# Probability and Some Statistics

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# Lecture 10 - Outline

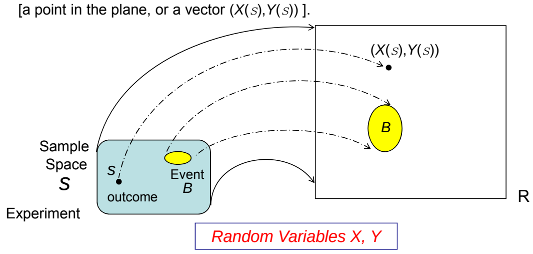
1. **Pairs of Random Variables**
   - **CDFs: joint, marginal, conditional, independence**
   - **PMFs: joint, marginal, conditional, independence**



# Chapter 4 **Bivariate Random Variables**

Two RVs $X$ and $Y$ defined on the same sample space $S$ are called bivariate RVs, Joint RVs, or random vectors (of length 2)

For each outcome $s \in S$ of the random experiment, $X(s)$ and $Y(s)$ are real numbers [a point in the plane, or a vector $(X(s),Y(s))$ ].

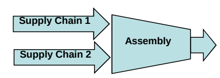



$P[\text{Event } B] = P_{X,Y}(B)$

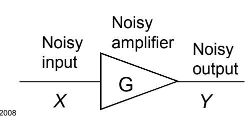


*Examples:*

- Time and duration of telephone calls.
  - Times may be originated at random
  - Calls at times when the rates are cheaper may be longer
- Work-in-process across two supply chains feeding an assembly operation




- Complex random signal
  - Amplitude and phase
  - Real and imaginary part
- Input and output of a noisy amplifier




$Y = GX+n$



## Statistical Description of Bivariate RVs $X$ and $Y$
## Summary

| | CDF | PMF (discrete) PDF (continuous) | Moments |
|---|---|---|---|
| Joint | $F_{X,Y}(x,y)$ | $P_{X,Y}(x,y)$ <br> $f_{X,Y}(x,y)$ | $E[XY]$ Correlation <br> $\text{Cov}(X,Y)$ Covariance <br> $=E[XY] - E[X]E[Y]$ |
| Marginal | $F_X(x)$, $F_Y(y)$ | $P_X(x)$, $P_Y(y)$ <br> $f_X(x)$, $f_Y(y)$ | $E[X]$, $E[Y]$ Means <br> $\text{Var}[X] = E[X^2] - (E[X])^2$ <br> $\text{Var}[Y] = E[Y^2] - (E[Y])^2$ <br> Variance |
| Conditional | $F_{X\|Y}(x \mid y)$ | $P_{X\|Y}(x \mid y)$ <br> $f_{X\|Y}(x \mid y)$ | $E[X \mid y]$ Conditional expectation |

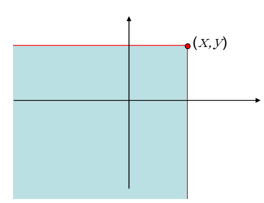


## 4.1 Joint CDF of Bivariate RVs

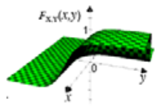
Definition

$$F_{X,Y}(x,y) = P[\{X \le x\} \cap \{Y \le y\}]$$
$$= P[\{s \in S : X(s) \le x \text{ and } Y(s) \le y\}]$$
$$\equiv P[X \le x, Y \le y]$$

= Probability that random point lies in shaded area



## Properties




- $0 \le F_{X,Y}(x,y) \le 1$ for all $x$ and $y$
- $F_{X,Y}(x,y)$ is nondecreasing in $x$ for fixed $y$
- $F_{X,Y}(x,y)$ is nondecreasing in $y$ for fixed $x$
- $\lim_{x,y\to\infty} F_{X,Y}(x,y) \equiv F_{X,Y}(\infty,\infty) = 1$

  $\lim_{x,y\to-\infty} F_{X,Y}(x,y) \equiv F_{X,Y}(-\infty,-\infty) = 0$
- For any $x,y$, $\lim_{x\to-\infty} F_{X,Y}(x,y) \equiv F_{X,Y}(-\infty,y) = 0$

  $\lim_{y\to-\infty} F_{X,Y}(x,y) \equiv F_{X,Y}(x,-\infty) = 0$
- $F_{X,Y}(x,y)$ is a right-continuous as a function of $x$ and right continuous as a function of $y$

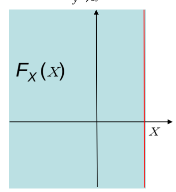
$$\lim_{\epsilon>0,\epsilon\to 0} F_{X,Y}(x+\epsilon,y) = F_{X,Y}(x,y)$$
$$\lim_{\epsilon>0,\epsilon\to 0} F_{X,Y}(x,y+\epsilon) = F_{X,Y}(x,y)$$

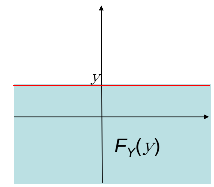


## Marginal CDFs

Individual CDFs of $X$, $Y$, called the marginal CDFs, can be obtained from $F_{X,Y}(x,y)$. For any $x,y$

$$\lim_{x\to\infty} F_{X,Y}(x,y) \equiv F_{X,Y}(\infty,y)$$
$$= P[X \le \infty, Y \le y] = P[Y \le y] = F_Y(y)$$
$$\lim_{y\to\infty} F_{X,Y}(x,y) \equiv F_{X,Y}(x,\infty) = F_X(x)$$



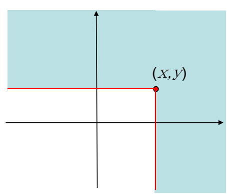




$F_X(x) = F_{X,Y}(x,\infty)$ $F_Y(y) = F_{X,Y}(\infty,y)$

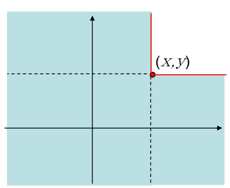


## Probabilities from CDF

$$P[\{X > x\} \cup \{Y > y\}] = 1 - F_{X,Y}(x,y)$$

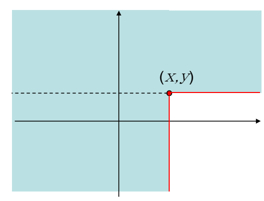



$$P[\{X \le x\} \cup \{Y \le y\}] = F_X(x) + F_Y(y) - F_{X,Y}(x,y)$$
$$= F_{X,Y}(x,\infty) + F_{X,Y}(\infty,y) - F_{X,Y}(x,y)$$

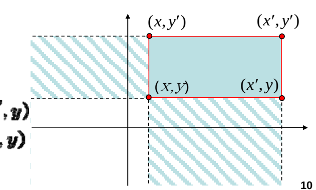





$$P[\{X \le x\} \cup \{Y > y\}] = F_{X,Y}(x,y) + (1 - F_Y(y)) = F_{X,Y}(x,y) + (1 - F_{X,Y}(\infty,y))$$




$$P\left[X \in (x,x'], Y \in (y,y']\right] = F_{X,Y}(x',y') - F_{X,Y}(x',y) - F_{X,Y}(x,y') + F_{X,Y}(x,y)$$






## Joint CDF of Independent RVs

> *Recall*
> Two events $A$, $B$ are said to be *independent* if and only if $P[A \cap B] = P[A]\,P[B]$
> Implies $P[B|A] = P[B]$, $P[A|B] = P[A]$

The RVs $X$ and $Y$ are said to be mutually independent if every event $A$ defined by $X$ is independent of every event $B$ defined by $Y$.

$$A = \{X \le x\}; \quad B = \{Y \le y\}$$
$$P[A,B] \equiv F_{X,Y}(x,y) = P[A]P[B] \quad \text{Independence}$$
$$\boxed{F_{X,Y}(x,y) = F_X(x)F_Y(y)}$$

The joint CDF of independent RVs factors into a product of the marginals



If the RVs are independent, the joint CDF $F_{X,Y}(x,y)$ may be determined from the marginal CDFs $F_X(x)$ and $F_Y(y)$.

If the RVs are not independent, the joint CDF may not be determined from the marginal CDFs. Many different CDFs may have the same marginals! Many different surfaces $F_{X,Y}(x,y)$ may have the same limit values $F_{X,Y}(x,\infty)$ $F_{X,Y}(\infty,y)$

## 4.2 Joint PMF (Discrete Bivariate RVs)

### Definition

For discrete RVs $X$ and $Y$ the joint PMF

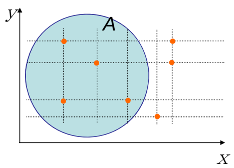
$$P_{X,Y}(x, y) = P[X = x \text{ and } Y = y]$$

describes a collection of point masses in the plane

$$\sum_x \sum_y P_{X,Y}(x, y) = 1$$

### Probabilities from Joint PMF

$P[(X,Y) \in A]$ = sum of all probability masses in the region $A$

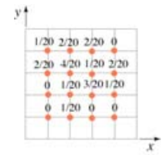


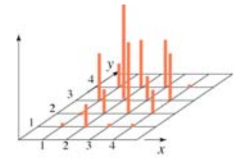

This also holds if $A$ is a curve (including a straight line as a special case) — just sum up all the probability masses on the curve



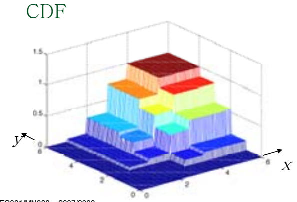
## Example

PMF




CDF




```
X = linspace(0,6); Y = X;
P = [ 0 0 .1 .05 0; .05 0.05 0.2 0.1 0; 0 0.15 0.05 0.1
0; 0 0.05 0.1 0 0; 0 0 0 0 0];
S1 = zeros(100,100);
 for i = 1:5
   for j = 1:5
     mask1 = X >= i;
     mask2 = Y >= j;
     S1(mask1, mask2) = S1(mask1, mask2) + P(i,j);
   end
end
mesh(X,Y,S1);
```



## Marginal PMFs

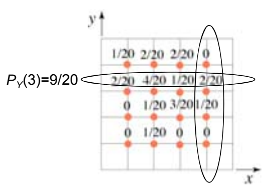
$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

The marginal PMFs of $X$ and $Y$ may be obtained from the joint PMF by projection (summing) in the $y$ and $x$ directions, respectively.

The joint PMF may not be determined from the marginals (unless the variables are independent)

### Example






## Conditional PMF

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

$$P_{X|Y}(x|y) = P[X = x | Y = y]$$
$$P_{X,Y}(x,y) = P[X = x \cap Y = y]$$
$$P_Y(y) = P[Y = y]$$

= Conditional probability mass function of $X$, given that the outcome associated with event $\{Y = y\}$ was observed,

$$\text{Conditional} = \frac{\text{Joint}}{\text{Marginal}}$$

Properties: $P_{X|Y}(x \mid y)$ is a PMF on $X$. It depends on $y$, but it is not a PMF on $Y$

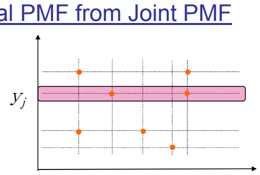
$\Sigma_x P_{X|Y}(x \mid y) = 1$

$P_{X|Y}(x \mid y)$ in [0,1]



## Calculation of Conditional PMF from Joint PMF

$$P_{X|Y}(x|y_j) = \frac{P_{X,Y}(x,y_j)}{P_Y(y_j)}$$




$P_Y(y_j)$ = sum of all probability masses lying at $y = y_j$

$P_{X|Y}(x|y_j)$ = ratio of the probability mass at $x$ to the total mass on the line $y = y_j$



## Total probability theorem for conditional PMFs

Since the events $\{Y = y_1\}$, $\{Y = y_2\}$, …, $\{Y = y_n\}$, … are mutually disjoint and collectively exhaustive, forming a partition of the sample space $S$,

$$P_X(x) = \sum_{y_i \in S_Y} P_{X|Y}(x|y_i) P_Y(y_i)$$

### Bayes' Rule applied to PMFs

Bayes' rule: $P(A|B) = P(B|A)\, P(A) / P(B)$

$$P_X(x_i) = \sum_{y_j \in S_Y} P_{X,Y}(x_i, y_j) = \sum_j P_{X|Y}(x_i|y_j) P_Y(y_j)$$

$$P_{Y|X}(y_j|x_i) = \frac{P_{X,Y}(x_i,y_j)}{P_X(x_i)} = \frac{P_{X|Y}(x_i|y_j) P_Y(y_j)}{\sum_{j'} P_{X|Y}(x_i|y_{j'}) P_Y(y_{j'})}$$

## Independence

$X$, $Y$ are said to be *independent* if and only if

$$P_{X,Y}(x,y) = P_X(x)\, P_Y(y)$$

i.e., joint PMF factors into product of marginals. This factorization also follows from factorization of the joint CDF

Or

$X$,$Y$ are said to be independent if and only if

$P_{X|Y}(x|y) = P_X(x)$

and $P_{Y|X}(y|x) = P_Y(y)$



## Examples of Dependent and Independent RVs:

Experiment 1: roll two 6-sided dice

$X(s)$ = value of first die

$Y(s)$ = 7 – value of first die

Experiment 2: roll two 6-sided dice

$X(s)$ = value of first die

$Y(s)$ = value of second die

Both experiments have the same range of outcomes

$X(s)$, $Y(s)$ take values in 1…6

$P_X(n)$ = 1/6 for $n$ = 1, …, 6 in both cases!

$P_Y(n)$ = 1/6 for $n$ = 1, …, 6 also in both!!!

However, the two experiments are clearly different!

- In Experiment 1, when $X = n$, $Y = 7 - n$ → very dependent!
- In Experiment 2, when $X = n$, $Y$ is totally random → very independent!
- Knowing $P_X()$, $P_Y()$ is insufficient to judge dependence or independence!



## *Example:* Embedded RVs

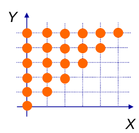
$Y$ is a Poisson RV with parameter $\lambda$

Conditioned on $Y = n$, $X$ is a binomial RV with parameters ($n$, $p$)

Determine the Marginal PMF of $X$

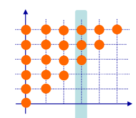
$$P_{X|Y}(k|n) = \binom{n}{k} p^k (1-p)^{n-k},\ k = 0, \ldots, n$$

- Conditioned on $Y = n$, $X$ has values from 0 to $n$.
- Conditioned on $X = k$, $Y$ has values from $k$ to $\infty$
- Since $X \le Y$ the joint PMF of $X$ and $Y$ is nonzero on a triangular region






## Marginal PMF of $X$



$$\begin{aligned} P_X(3) &= \sum_n I\{n \ge 3\} \binom{n}{3} p^3 (1-p)^{n-3} \frac{\lambda^n}{n!} e^{-\lambda} \\ &= \sum_{n=3}^{\infty} \frac{n!}{3!(n-3)!} p^3 (1-p)^{n-3} e^{-\lambda} \frac{\lambda^n}{n!} \\ &= \frac{(\lambda p)^3}{3!} e^{-\lambda} \sum_{n=3}^{\infty} \frac{(\lambda(1-p))^{n-3}}{(n-3)!} \\ &= \frac{(\lambda p)^3}{3!} e^{-\lambda p} \end{aligned}$$

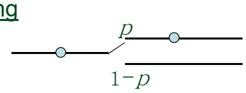
$X$ is Poisson with parameter $\lambda\, p$

Since the Binomial is the sum of $n$ Bernoulli RVs, the sum of a Poisson random number of Bernoulli RVs is Poisson. Poisson RV remains Poisson under independent sampling.



## Application: Random Routing

Messages going into a router are sent along path 1 with probability $p$, path 2 with probability $1 - p$




If one receives a Poisson number of messages $n$ with rate $\lambda$:

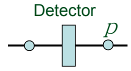
- the number of messages sent on path 1 is Poisson with rate $p\lambda$,
- the number of messages sent on path 2 is Poisson with rate $(1 - p)\lambda$

## Application: Geiger counter with finite efficiency

$\alpha$-particles are counted in a Geiger counter

Each particle is detected with probability $p$ and not detected with probability $1 - p$. Detections are independent of each other

If $n$ $\alpha$-particles are emitted (Poisson number), the number detected (the count in the Geiger counter) is a binomial RV with parameters ($n$, $p$)






b) Assume $X$ was observed and found to be $X = 3$. Compute $P_{Y|X}(n|3)$

Bayes' rule

$$\begin{aligned} P_{Y|X}(n|3) &= \frac{p_{X|Y}(3|Y=n)\, p_Y(n)}{\sum_{y'} p_{X|Y}(3|y_{y'})\, p_Y(y_{y'})} \\ &= \frac{I\{n \ge 3\} \binom{n}{3} p^3 (1-p)^{n-3} \frac{\lambda^n}{n!} e^{-\lambda}}{p_X(3)} \\ &= \frac{I\{n \ge 3\} \binom{n}{3} p^3 (1-p)^{n-3} \frac{\lambda^n}{n!} e^{-\lambda}}{\frac{(\lambda p)^3}{3!} e^{-\lambda p}} \\ &= I\{n \ge 3\} e^{-\lambda(1-p)} \frac{(\lambda(1-p))^{n-3}}{(n-3)!} \end{aligned}$$

Conditioned on $X = 3$, $Y = 3$ + Poisson with parameter $\lambda(1 - p)$. This is called a shifted Poisson.