EC381/MN308
Probability and Some Statistics

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Lecture 12 - Outline

1. **Pairs of Random Variables**
   - **Functions of 2 random variables**
   - **Covariance**
   - **Conditional expectations**

4.4 Functions of Two RVs

The function $Z = g(X,Y)$ defines a new RV Z on same probability space of X, Y

Examples: $Z = X + Y$, $Z = XY$, $Z = X^2 + Y^2$, $Z = aX + bY$

Given the joint PMF or PDF of X, Y, determine the statistical properties of Z:

Discrete: $P_Z(z) \equiv P(Z = z) = \sum_{x,y:G(x,y)=z} P_{X,Y}(x,y)$

Continuous: use CDF, not PDF:

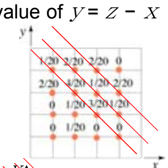
$$F_Z(z) \equiv P(Z \le z) = \int_{x,y:G(x,y)\le z} f_{X,Y}(x,y)dxdy$$

$$f_Z(z) = \frac{d}{dz}F_Z(z)$$

Special Case. Sum of Two RVs: $Z = X + Y$

Discrete: For each x, there is a unique value of $y = z - x$

$$P_Z(z) = \sum_{x \in S_X} p_{X,Y}(x, z-x)$$



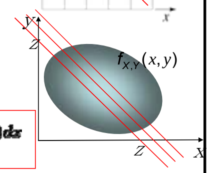
Continuous:

$$F_Z(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} f_{X,Y}(x,y)dy$$

Switch variables of integration to x, $z' = x + y$

$$F_Z(z) = \int_{-\infty}^{z} dz' \int_{-\infty}^{\infty} f_{X,Y}(x, z'-x)dx$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)dx$$



Example 1

Customers at a bank
- Each customer has random number of transactions N
- Each transaction takes random time T

Joint PMF of (N,T) is represented as matrix

If the total time per customer is defined as a RV $Z = NT$ find $P_Z(z)$

$P_{N,T}(N = n, T = t)$	$T = 25$	$T = 50$
$N = 1$	0.1	0.2
$N = 2$	0.2	0.2
$N = 3$	0.1	0.2

Possible values of $Z = NT$:
 25, 50, 75, 100, 150

Inspection of table yields:
 $P_z(25) = 0.1$; $P_z(50) = 0.2 + 0.2 = 0.4$
 $P_z(75) = 0.1$; $P_z(100) = 0.2$; $P_z(150) = 0.2$

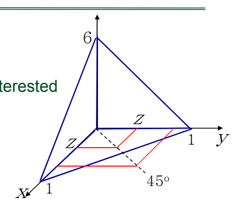
Example 2

$$f_{X,Y}(x,y) = 6(1-x-y),$$
$$x, y \ge 0, x + y \le 1$$

$Z = \max(X,Y)$ is the RV in which we are interested

Find $f_Z(z)$

$F_Z(z) = P[\max(X,Y) < z]$

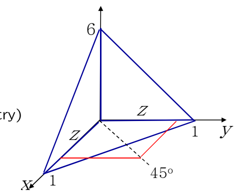


First do region $z < 0.5$:

$$F_Z(z) = \int_0^z \int_0^z f_{X,Y}(x,y)dxdy$$
$$= \int_0^z \int_0^z 6(1-x-y)dxdy$$
$$= \int_0^z 6(1-x)z - 3z^2$$
$$= 6z^2 - 6z^3, z \in (0, 0.5)$$

Now do region $z > 0.5$:

– Hard to define area
– Easier to define complement!
– Limits x: $z \to 1$, y: $0 \to 1-x$
:



$$F_Z(z) = 1 - 2\int_z^1 \int_0^{1-x} f_{X,Y}(x,y)\,dx\,dy \text{ (symmetry)}$$
$$= 1 - 2\int_z^1 \int_0^{1-x} 6(1-x-y)\,dx\,dy$$
$$= 1 - 2\int_z^1 3(1-x)^2\,dx$$
$$= 1 - 2(1-z)^3, \quad z \in (0.5, 1)$$

Sanity Check: Values must agree at 0.5

– $F_Z(z)$ is continuous for continuous RV

$$F_Z(z) = \begin{cases} 6z^2 - 6z^3 & z \in (0, 0.5) \\ 1 - 2[1-z]^3 & z \in (0.5, 1) \end{cases}$$

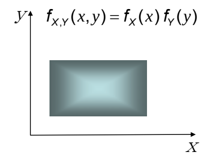
– They agree (both are 3/4); also $F_Z(1) = 1$ and $F_Z(0) = 0$. So we have

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} 12z - 18z^2 & z \in (0, 0.5) \\ 6(1-z)^2 & z \in (0.5, 1) \end{cases}$$

Special Case. Sum of Two Independent RVs: $Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

Convolution



PDF of the sum of two independent RVs = convolution of their marginal PDFs

Generalizes to n independent RVs by induction

Example: Convolution of exponential PDFs yields Erlang (gamma) PDF

Expectation of Derived RVs

$Z = g(X,Y)$

– Can compute expectation without computing $f_Z(z)$

– If X, Y are jointly discrete:

$$E[Z] = \sum_{x \in S_x} \sum_{y \in S_y} g(x,y)\, p_{X,Y}(x,y)$$

– If X, Y are jointly continuous,

$$E[Z] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$$

• Just integrate …

$E[\bullet]$ is a linear operator

$E[g_1(X,Y) + \ldots + g_N(X,Y)] = E[g_1(X,Y)] + \ldots + E[g_N(X,Y)]$

$E[X + Y] = E[X] + E[Y]$

$E[aX + bY] = aE[X] + bE[Y]$

True for discrete, continuous, mixed.

Proof: Can exchange integration, summation order if X,Y jointly continuous, or summation order if X, Y jointly discrete.

4.5 Covariance and Cross-Correlation

Mean and Variance of the Sum of Two RVs

$Z = X + Y$

Mean: $E[Z] = E[X] + E[Y]$

Variance:

$$\text{Var}(Z) = E[Z^2] - E[Z]^2$$
$$= E[X^2 + 2XY + Y^2] - E[X]^2 - 2E[X]E[Y] - E[Y]^2$$
$$= (E[X^2] - E[X]^2) + (E[Y^2] - E[Y]^2)$$
$$+ 2(E[XY] - E[X]E[Y])$$
$$= \text{Var}(X) + \text{Var}(Y) + 2(\underbrace{E[XY]}_{\text{correlation}} - E[X]E[Y])$$

covariance

$$E[XY] - E[X]E[Y] = E[(X - E[X])(Y - E[Y])]$$

Correlation: $r_{X,Y} \equiv E[XY]$

Covariance:

$$\text{Cov}[X,Y] \equiv E[XY] - E[X]E[Y] = E[(X - E[X])(Y - E[Y])]$$

Identities:

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X,Y]$$
$$\text{Cov}[X,Y] = r_{X,Y} - E[X]E[Y]$$
$$X = Y \Rightarrow \text{Cov}[X,Y] = \text{Var}[X],\ r_{X,Y} = E[X^2]$$

Correlation Coefficient of X, Y:

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}} = \frac{\text{Cov}[X,Y]}{\sigma_X \sigma_Y}$$

Captures dependency of random variations in X and in Y

Key property: $|\rho_{X,Y}| \le 1$ or $-1 \le \rho_{X,Y} \le 1$

Proof of the Bounds $|\rho_{X,Y}| \le 1$ or $-1 \le \rho_{X,Y} \le 1$

Cauchy-Schwarz inequalities

$$(\int_{-\infty}^{\infty} f(x)g(x)dx)^2 \le (\int_{-\infty}^{\infty} f^2(x)dx)(\int_{-\infty}^{\infty} g^2(x)dx)$$

$$(\sum_i x_i y_i)^2 \le (\sum_i x_i)^2 (\sum_i y_i)^2$$

$$\begin{aligned}
Cov[X,Y]^2 &= (\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-E[x])(y-E[Y])f_{X,Y}(x,y)dxdy)^2 \\
&= [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(x-E[X])f_{X,Y}(x,y)^{1/2}\}\{(y-E[Y])f_{X,Y}(x,y)^{1/2}\}dxdy]^2 \\
&\le [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-E[X])^2 f_{X,Y}(x,y)dydx][\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(y-E[Y])^2 f_{X,Y}(x,y)dxdy] \\
&\le [\int_{-\infty}^{\infty}(x-E[X])^2 f_X(x)dx][\int_{-\infty}^{\infty}(y-E[Y])^2] f_Y(y)dy \\
&\le Var[X] \quad Var[Y]
\end{aligned}$$

$$\Rightarrow \rho(X,Y) = \frac{Cov[X,Y]}{\sqrt{Var[X]Var[Y]}} \in (-1,1)$$

Orthogonal RVs:

X and Y are said to be orthogonal if $E[XY] = 0$
If X and Y are orthogonal, then $E[(X + Y)^2] = E[X^2] + E[Y^2]$
Second moment of sum = Sum of second moments

Uncorrelated RVs:

X and Y are said to be uncorrelated if Cov$[X,Y] = 0$ or $\rho_{X,Y} = 0$
If X and Y are uncorrelated, then Var$[X + Y]$ = Var$[X]$ + Var$[Y]$
Variance of sum = Sum of variances
If X and Y are uncorrelated, then aX and bY are uncorrelated

Completely Correlated RVs

X and Y are said to be completely correlated if $|\rho_{X,Y}| = 1$
If $Y = aX + b$, then X and Y are completely correlated

$$\rho_{X,Y} = \begin{cases} -1 & a < 0 \\ 0 & a = 0 \\ 1 & a > 0 \end{cases}$$

Example 1

$$f_{X,Y}(x,y) = \begin{cases} xy & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \int_0^2 xydy = 2x, \ x \in (0,1)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_0^1 xydy = \frac{y}{2}, y \in (0,2)$$

$$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^1 2x^2 dx = 2/3$$

$$E[Y] = \int_{-\infty}^{\infty} yf_Y(y)du = \int_0^2 \frac{y^2}{2}dy = 4/3$$

$$\text{Var}[X] = E[X^2] - E[X]^2 = \int_0^1 2x^3 dx - 4/9 = 1/2 - 4/9 = \frac{1}{18}$$

$$\text{Var}[Y] = E[Y^2] - E[Y]^2 = \int_0^2 \frac{y^3}{2}dy - 16/9 = 2 - 16/9 = \frac{2}{9}$$

Correlation:

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy = \int_0^2\int_0^1 x^2y^2dxdy = \frac{8}{9}$$

- Cov$[X,Y] = E[XY] - E[X]E[Y]$ = 8/9 – (2/3)(4/3) = 0
- $\rho_{X,Y} = 0$
- $E[X+Y]$ = 2/3 + 4/3 = 2
- Var$[X + Y]$ = Var$[X]$ + Var$[Y]$ + 2 Cov$[X,Y]$ = 5/18
- Var$[X - Y]$ = Var$[X]$ + Var$[Y]$ – 2 Cov$[X,Y]$ = 5/18
- X,Y ***are*** uncorrelated but they ***are not*** orthogonal since the correlation is 8/9 but the covariance is 0.

Covariance Matrix

- Form vector $\underline{X} = \begin{pmatrix} X \\ Y \end{pmatrix}$
- Expected value $E[\underline{X}] = \begin{pmatrix} E[X] \\ E[Y] \end{pmatrix} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$
- Covariance matrix

TRANSPOSE

$$\begin{aligned}
\Sigma_{\underline{X}} &= E[(\underline{X} - E[\underline{X}])(\underline{X} - E[\underline{X}])^T] \\
&= \begin{pmatrix} \sigma_X^2 & \text{Cov}[X,Y] \\ \text{Cov}[X,Y] & \sigma_Y^2 \end{pmatrix}
\end{aligned}$$

Independent vs Uncorrelated RVs

- If X, Y are uncorrelated, then
  - Cov$[X,Y] = 0$
  - $E[XY] = E[X]\,E[Y]$
  - Var$[X + Y]$ = Var$[X]$ + Var$[Y]$
- If X, Y are independent RVs, they must be uncorrelated, i.e.,
  - Cov$[X,Y] = 0$
  - $E[XY] = E[X]\,E[Y]$
  - Var$[X+Y]$ = Var$[X]$ + Var$[Y]$

  and, in addition,
  - $E[g(X)\,h(Y)] = E[g(X)]\,E[h(Y)]$
  - $E[X|Y=y] = E[X]$ for all y
  - $E[Y|X=x] = E[Y]$ for all x
- If X, Y are uncorrelated, they need not be independent

Example

- X, Y Independent?
  - NO! 0s in row

$P_{X,Y}(x,y)$	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	0.01	0	0
$X = 1$	0.09	0.09	0
$X = 2$	0	0	0.81

- X, Y independent?
  - Yes. Can factor into

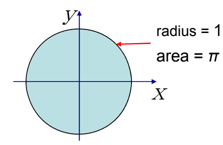
$P_X(0) = 0.6$, $P_X(1) = 0.4$,

$P_Y(0) = 0.1$, $P_Y(1) = 0.3$,
$P_Y(2) = 0.4$, $P_Y(3) = 0.2$

$P_{X,Y}(x,y)$	$Y = 0$ 0.1	$Y = 1$ 0.3	$Y = 2$ 0.4	$Y = 3$ 0.2
$X = 0$ 0.6	0.06	0.18	0.24	0.12
$X = 1$ 0.4	0.04	0.12	0.16	0.08

Example

X,Y be uniformly distributed on the unit disc



X,Y are not independent (since PDF is not a rectangle).
However, $E[XY] = 0$ (see below) and $E[X] = 0$, $E[Y] = 0$, so that X,Y are orthogonal and uncorrelated!

$$E[XY] = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} xy\,dy\,dx$$
$$= \int_{-1}^{1} 0 = 0$$

Example

- Independent? $f_{X,Y}(x,y) = \begin{cases} xy & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$
  - Yes!
  - Rectangular region
  - Factors
- X, Y independent, identically distributed (i.i.d.)

$$f_X(x) = \begin{cases} 1 - x/2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$
$$f_{X,Y}(x,y) = \begin{cases} (1 - x/2)(1 - y/2) & 0 \le x, y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

- Define $Z = \max(X,Y)$. Find $f_Z(z)$
- Approach: first, find $F_Z(z)$, then differentiate

$$F_Z(z) = \int_{-\infty}^{z}\int_{-\infty}^{z} f_{X,Y}(x,y)\,dx\,dy$$
$$= \begin{cases} \int_0^z \int_0^z (1 - x/2)(1 - y/2)\,dx\,dy & 0 \le z \le 2 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} (z - z^2/4)^2 & 0 \le z \le 2 \\ 0 & \text{otherwise} \end{cases}$$
$$f_Z(z) = \begin{cases} 2(z - z^2/4)(1 - z/2) & 0 \le z \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Conditional Expected Value

Determine the expected value of $Z = G(X,Y)$ given the observed value $Y = y$:

$$E\left[G(X,Y) \middle| Y = y\right]$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx}$$
$$E[G(X,Y)|Y = y] = \int_{-\infty}^{\infty} G(x,y) f_{X|Y}(x|y)\,dx$$

Conditional Statistics

$$E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$$
$$\text{Var}[X|Y = y] = E[X^2|Y] - (E[X|Y])^2$$
$$\int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\,dx - (E[X|Y])^2$$

Iterated expectation

$$E[X] = E\{E[X|Y = y]\}$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X|Y}(x|y) f_Y(y)\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx\right\} f_Y(y)\,dy$$

In general, $E[g(X)] = E\{E[g(X)|Y = y]\}$

Example 1

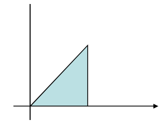
- *X* uniform in [0,1]; given *X* = x, select *Y* uniform in [0, x]

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x}, & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

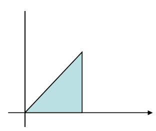
$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = \begin{cases} \frac{1}{x} & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \int_y^1 \frac{1}{x} dx = -\ln y, \quad y \in (0,1)$$



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \text{ if } f_Y(y) > 0$$
$$= \begin{cases} \frac{-1}{x \ln y} & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$
$$= \int_y^1 x \frac{-1}{x \ln y} dx, \ y \in (0,1)$$
$$= \frac{1-y}{-\ln y}, \ y \in (0,1)$$



$$E[X^2|Y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y) dx$$
$$= \int_y^1 x^2 \frac{-1}{x \ln y} dx, \ y \in (0,1)$$
$$= \frac{1-y^2}{-2\ln y}, \ y \in (0,1)$$

$$\text{Var}[X|Y=y] = \frac{1-y^2}{-2\ln y} - \frac{(1-y)^2}{(\ln y)^2}, \ y \in (0,1)$$

$$E\{E[X|Y=y]\} = \int_0^1 \frac{1-y}{-\ln y}(-\ln y)dy$$
$$= \int_0^1 (1-y)dy = 0.5 = E[X]$$