EC381/MN308
Probability and Some Statistics

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Lecture 13 - Outline

1. **Pairs of Random Variables**
   - **Jointly Gaussian random variables**
   - **Covariance matrices**

4.6 Jointly Gaussian RVs

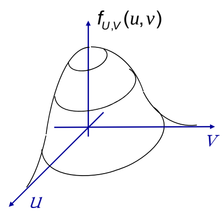
A. Two Independent Gaussian RVs

i) Two independent unit Gaussian RVs *U*, *V*

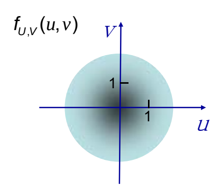
N(0,1) $\quad f_V(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \quad f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$

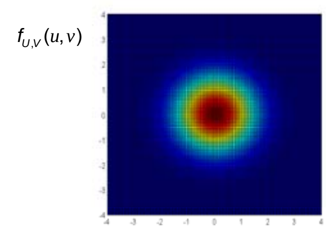
Joint PDF

$$f_{U,V}(u,v) = f_U(u) f_V(v) = \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}}$$



This is a circularly symmetric PDF



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X = linspace(-4,4);
Y = normpdf(X,0,1);
surf(X,X,Y'*Y);
```


ii) Two independent non-unit Gaussian RVs *X, Y*

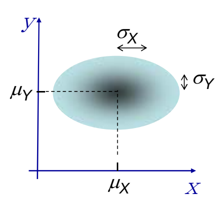
If $\quad X = \sigma_X U + \mu_X, \qquad Y = \sigma_Y V + \mu_Y$

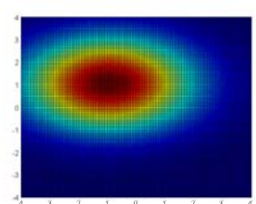
then $\quad X \sim N(\mu_X, \sigma_X^2), \qquad Y \sim N(\mu_Y, \sigma_Y^2)$

Since *U* and *V* are independent RVs, *X* and *Y* are also independent RVs :

Proof:
X depends only on *U*, *Y* depends only on *V*
Therefore, events defined by *X* are defined by *U*,
and events defined by *Y* are defined by *V*
and therefore *X*, *Y* are independent also!

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}}$$





```
X = linspace(-4,4);
Y1 = normpdf(X,1,sqrt(2));
Y2 = normpdf(X,-1,2);
surf(X,X,Y1'*Y2);
```

The curve $f_{X,Y}(x,y)$ = constant defines an ellipse that is centered at (μ_X, μ_Y) and whose axes are parallel to the coordinate axes (this is true since the RVs are independent).

iii) Sum of Two Independent Gaussians RVs

$X \sim N(\mu_X, \sigma_X^2)$,

$Y \sim N(\mu_Y, \sigma_Y^2)$

then $Z = X + Y \sim N(\mu_Z, \sigma_Z)$ where $\mu_Z = \mu_X + \mu_Y$

$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$

Proof:

1. The mean of the sum is always equal to the sum of the means.
2. Since X and Y are independent, they must be uncorrelated, i.e., $\text{cov}[X, Y] = 0$. Therefore the variance of the sum equals the sum of the variances.
3. Since X and Y are independent, the PDF of the sum equals the convolution of the PDFs. Since the convolution of two Gaussian functions is again a Gaussian function, Z must be Gaussian.

Proof that the convolution of two Gaussian functions is a Gaussian function

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)dx = \int_{-\infty}^{\infty} Ce^{-\frac{(x)^2}{2\sigma_X^2}-\frac{(z-x)^2}{2\sigma_Y^2}}dx$$

$$= Ce^{-\frac{z^2}{2\sigma_Y^2}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}(\frac{1}{\sigma_X^2}+\frac{1}{\sigma_Y^2})+\frac{xz}{\sigma_Y^2}}dx$$

$$= Ce^{-\frac{z^2}{2\sigma_Y^2}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}(\frac{1}{\sigma_X^2}+\frac{1}{\sigma_Y^2})+\frac{xz}{\sigma_Y^2}-\frac{z^2}{2}\frac{\sigma_X^2}{\sigma_Y^2(\sigma_X^2+\sigma_Y^2)}+\frac{z^2}{2}\frac{\sigma_X^2}{\sigma_Y^2(\sigma_X^2+\sigma_Y^2)}}dx$$

$$= Ce^{-\frac{z^2}{2\sigma_Y^2}+\frac{z^2}{2}\frac{\sigma_X^2}{\sigma_Y^2(\sigma_X^2+\sigma_Y^2)}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}(\frac{1}{\sigma_X^2}+\frac{1}{\sigma_Y^2})+\frac{xz}{\sigma_Y^2}-\frac{z^2}{2}\frac{\sigma_X^2}{\sigma_Y^2(\sigma_X^2+\sigma_Y^2)}}dx$$

$$= Ce^{-\frac{z^2}{2(\sigma_X^2+\sigma_Y^2)}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{1}{\sigma_X^2}+\frac{1}{\sigma_Y^2})(x-\frac{\sigma_X^2}{\sigma_X^2+\sigma_Y^2}z)^2}dx$$

$$= C_1e^{-\frac{z^2}{2(\sigma_X^2+\sigma_Y^2)}}$$

B. Jointly Gaussian RVs (Not Necessarily Independent)

Generally, if U and V are independent unit Gaussian RVs then

$$X = aU + bV + \mu_X$$

and

$$Y = cU + dV + \mu_Y$$

are called jointly Gaussian RVs.

X and Y have Gaussian marginal PDFs with means μ_X and μ_Y and variances σ_X and σ_Y that depend on a, b, c, and d, but X and Y are, in general, correlated. Their correlation coefficient depends on the coefficients a, b, c, and d.

$U \sim N(0,1)$
$V \sim N(0,1)$
U and V independent

$X = aU + bV + \mu_X$
$Y = cU + dV + \mu_Y$

$E[X] = \mu_X$, $E[Y] = \mu_Y$

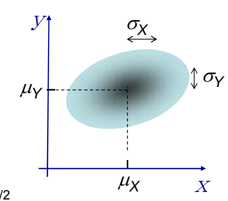
$\sigma_X^2 = a^2 + b^2$, $\sigma_Y^2 = c^2 + d^2$

$\text{Cov}[X,Y] = \text{Cov}[aU + bV, cU + dV]$

$= \text{Cov}[aU, cU] + \text{Cov}[aU, dV] + \text{Cov}[bV, cU] + \text{Cov}[bV, dV]$

$= ac + bd$

$\rho_{X,Y} = (ac + bd) / [(a^2 + b^2)(c^2 + d^2)]^{1/2}$



General expression for the joint PDF of two Gaussian RVs

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right]$$

where ρ is the correlation coefficient of X and Y and $|\rho| \le 1$.

Characterized by five parameters: μ_X, μ_Y, σ_X, σ_Y, $\rho_{X,Y}$
Joint PDF can be expressed as exponential of negative quadratic ($Q(x, y) < 0$ for all x, y)

$$f_{X,Y}(x,y) = Ce^{Q(x-\mu_X,\ y-\mu_Y)}$$

$$Q(x,y) = -\frac{1}{2(1-\rho)^2}\left(\left(\frac{x}{\sigma_X}\right)^2 - 2\rho\frac{x}{\sigma_X}\frac{y}{\sigma_Y} + \left(\frac{y}{\sigma_Y}\right)^2\right)$$

$$C = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

Property 1

The marginal PDFs of jointly Gaussian X and Y are Gaussian: $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$. They do not depend on ρ.

Property 2

If $\rho = 0$, then C and $Q(u,v)$ factor into

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}\sigma_X\sqrt{2\pi}\sigma_Y}\exp\left[-\frac{x^2}{2\sigma_X^2} - \frac{y^2}{2\sigma_Y^2}\right]$$

$$= f_X(x)\ f_Y(y)$$

Therefore, uncorrelated jointly Gaussian RVs are independent!
In general, uncorrelated RVs are not independent:

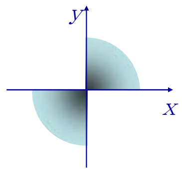
- Independent RVs are uncorrelated, but the reverse is generally not true
- However, if they are jointly Gaussian and uncorrelated, then they are also independent!

If jointly continuous random variables have Gaussian marginal PDFs, will their joint PDF be jointly Gaussian PDF? NO, as we now show.

Counter-example: Marginally Gaussian but not jointly Gaussian

Value of the marginal PDF is the area of the cross-section of the joint PDF surface

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{\pi}e^{-\frac{x^2+y^2}{2}} & xy\ge 0\\ 0 & xy<0\end{cases}$$



Double mass density in 1st & 3rd quadrants
But no mass density in 2nd & 4th quadrants

Property 3

If $|\rho|$ = 1, then Y must be a deterministic function of X

$Y = aX + b$ $b = (\mu_Y - a\mu_X)$, $a = \rho\,\sigma_Y/\sigma_X$

Still call this jointly Gaussian, but RVs are not jointly continuous (since PDF is zero outside of a line)

Property 4

$Z = aX + bY + c$ is Gaussian $Z \sim N(\mu_Z, \sigma_Z^2)$

Compute the PDF of Z:

$E[Z] = a\mu_X + b\mu_Y + c$,

$\mathrm{Var}[Z] = \sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho_{X,Y}\sigma_X\sigma_Y$

Property 5   Conditional probability density

The conditional PDF $f_{X|Y}(x|y)$ of jointly Gaussian RVs is also Gaussian with mean and variance

$$E[Y|X=x]=\mu_Y+\rho\frac{\sigma_X}{\sigma_Y}(x-\mu_X)$$

$$\mathrm{Var}[Y|X=x]=(1-\rho^2)\sigma_Y^2<\mathrm{Var}[Y]$$

Similarly,

$$E[X|Y=y]=\mu_Y+\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)$$

$$\mathrm{Var}[X|Y=y]=(1-\rho^2)\sigma_X^2$$

Proof:

$$Q(x,y)=-\frac{1}{2(1-\rho)^2}\left(\left(\frac{x}{\sigma_X}\right)^2-2\rho\frac{x}{\sigma_X}\frac{y}{\sigma_Y}+\left(\frac{y}{\sigma_Y}\right)^2\right)$$

$$=-\frac{1}{2}\left(\frac{x}{\sigma_X}\right)^2-\frac{1}{2(1-\rho)^2}\left(\left(\rho\frac{x}{\sigma_X}\right)^2-2\rho\frac{x}{\sigma_X}\frac{y}{\sigma_Y}+\left(\frac{y}{\sigma_Y}\right)^2\right)$$

$$=-\frac{1}{2}\left(\frac{x}{\sigma_X}\right)^2-\frac{1}{2(1-\rho)^2\sigma_Y^2}\left(y-\rho\frac{\sigma_Y}{\sigma_X}x\right)^2$$

$$C=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}=\frac{1}{\sqrt{2\pi}\sigma_X}\frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho^2}}$$

$$f_{X,Y}(x,y)=\frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}}\frac{1}{\sqrt{2\pi(1-\rho^2)}\sigma_Y}e^{-\frac{\left(y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)^2}{2(1-\rho^2)\sigma_Y^2}}$$

$$=f_X(x)f_{Y|X}(y|x)$$

$$f_{Y|X}(y|x)=\frac{1}{\sqrt{2\pi(1-\rho^2)}\sigma_Y}e^{-\frac{\left(y-\mu_Y-\rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right)^2}{2(1-\rho^2)\sigma_Y^2}}$$

Example

- Quiz 4.11
  - X,Y jointly Gaussian, ~N(0,1), correlation coefficient ρ = 0.5
  - Compute conditional PDF of X given Y = 2:

$$E[X|Y=y]=\mu_X+\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)$$

$$E[X|Y=2]=0+\rho(2-0)=1$$

$$\mathrm{Var}[X|Y=2]=(1-\rho^2)\sigma_X^2=\frac{3}{4}$$

  - Answer: $X|Y$=2 ~ N(1, 3/4)

Covariance Matrices

- The covariance matrix of (V, W) is the 2×2 identity matrix

$$\mathbf{X}=\begin{bmatrix}X_1\\X_2\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}Y_1\\Y_2\end{bmatrix}+\begin{bmatrix}b_1\\b_2\end{bmatrix}$$

$$=\mathbf{AY}+\mathbf{b}$$

$$\boldsymbol{\mu}_\mathbf{X}=\mathbf{A}\boldsymbol{\mu}_\mathbf{Y}+\mathbf{b}$$

- Covariance matrix of **X**

$$\boldsymbol{\Sigma}_\mathbf{X}=\mathbf{A}\boldsymbol{\Sigma}_\mathbf{Y}\mathbf{A}'$$