EC381/MN308
Probability and Some Statistics

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Lecture 3 - Outline

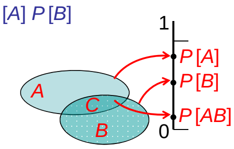
1. Independence.
2. Conditional probability.
3. Bayes' Theorem.

1.3 Two Events: *A*, *B*
Independence & Conditional Probability

A. Independence

Two events *A* and *B* are said to be independent if, and only if, the probability of the event $AB = A \cap B$ is the product of the individual probabilities:

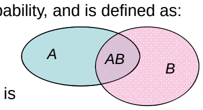
$$P[AB] = P[A]\,P[B]$$



B. Conditional Probability

The probability of observing an event *A*, given that another event *B* has been observed, is called the conditional probability, and is defined as:

$$P[A|B] = \frac{P[AB]}{P[B]}$$



assuming that $P[B] > 0$. The quantity $P[A|B]$ is called "the probability of *A* given *B*."

Similarly, $P[B|A] = \frac{P[BA]}{P[A]} = \frac{P[AB]}{P[A]}$

Therefore, $P[AB] = P[B|A]P[A] = P[A|B]P[B]$

If *A* and *B* are *independent*, then $P[A|B] = \frac{P[AB]}{P[B]} = \frac{P[A]P[B]}{P[B]} = P[A]$

so that $P[A|B] = P[A]$ i.e. it is irrelevant to *A* whether *B* is measured or not.

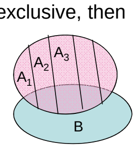
Properties of Conditional Probabilities:

i) $P[A|B] \in [0,1]$

ii) $P[A|A] = 1$

iii) If $A = A_1 \cup A_2 \cup \dots$ countable and mutually exclusive, then

$P[A|B] = P[A_1|B] + P[A_2|B] + \dots$



Proof:

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P\left[(\cup_{i=1}^{\infty} A_i) \cap B\right]}{P[B]}$$
$$= \frac{\sum_{i=1}^{\infty} P\left[A_i \cap B\right]}{P[B]}$$
$$= \sum_{i=1}^{\infty} \frac{P\left[A_i \cap B\right]}{P[B]} = \sum_{i=1}^{\infty} P[A_i|B]$$

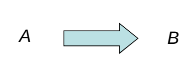
C. Bayes' Theorem

$$P[B|A] = \frac{P[A|B]P[B]}{P[A]}$$

Proof: $P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[A|B]P[B]}{P[A]}$

Bayes' Theorem provides a technique for evaluating the likelihood of cause, based on the observation of effect.

For example, for diagnosis, want to compute $P[B|A]$, what is the likelihood of cause, given the observed effect. (Use in quality control.)



Models based on Conditional Probability

Compound experiments:

e.g., *first* pick a factory, *then* observe the product;

first pick a test to perform, *then* generate the test result.

Conditional probability and Bayes' theorem provide a basis for generating a probability law, with the help of inference:

e.g., *first* observe the product, *then* determine the probability it was produced in a given factory.

Given an observed bit sequence 00010101 with possible errors, *then determine* the probability that the true bit sequence is 00110101?

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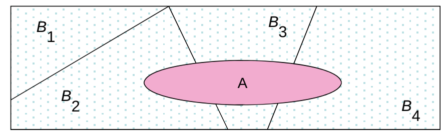
D. Total Probability Theorem

Given countable B_1, B_2, … , which are mutually exclusive and collectively exhaustive events, and event A:

$$P[A] = \sum_{i=1}^{\infty} P[A|B_i]P[B_i]$$

Proof: follows from the partition property for events.



Divide (that is, condition) and Conquer idea

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Example 1

Experiment: roll two 6-sided dice, all 36 outcomes equally likely

Events:

B: $\{(x,y)$: $\min(x,y) = 3\}$ (red triangles in figure)

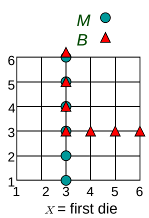
$M = \{(x,y)$: $x = 3\}$ (blue circles in figure)

Determine $P(M|B)$

$P(MB) = P[\{(3,3),(3,4),(3,5),(3,6)\}] = 4/36 = 1/9$

$P(B) = P[\{(3,3),(3,4),(3,5),(3,6),(4,3),(5,3),(6,3)\}] = 7/36$

$P(M|B) = P(MB)/P(B) = (1/9)/(7/36) = 4/7$



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Example 2

- You test positive for a disease.
- The test has a 5% false alarm rate and 0% misdetection.
- The unconditional probability for having the disease is 1/1000.

What is the probability you actually have the disease ?

$$P[\text{have disease} \mid \text{test } +]$$
$$= \frac{P[\text{have disease} \cap \text{test } +]}{P[\text{test } +]}$$
$$= \frac{P[\text{test } + \mid \text{have disease}]P[\text{have disease}]}{P[\text{test } + \cap \text{have disease}] + P[\text{test } + \cap \text{no disease}]}$$
$$= \frac{1 * 1/1000}{1/1000 + P[\text{test } + \mid \text{no disease}]P[\text{no disease}]}$$
$$= \frac{1/1000}{1/1000 + (5/100) * (1 - 1/1000)} = 0.0196$$

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Example 3: Parts from two factories

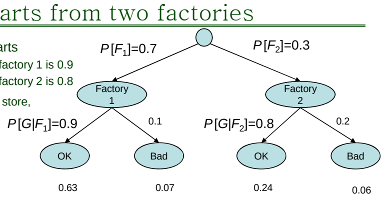
2 Factories, making "identical" parts

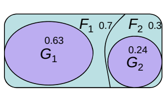
Probability that part is OK from factory 1 is 0.9

Probability that part is OK from factory 2 is 0.8

Factory 1 makes 70% of parts sold at store,

Factory 2 makes 30%





Q: What is the probability that, if the part you bought is OK, it was made in Factory 1?

Event G: Part is OK, $G = G_1 \cup G_2$

Event F_1 : Part made in Factory 1. Want $P(F_1|G)$

$$P[F_1|G] = \frac{P[F_1 \cap G]}{P[G]} = \frac{P[G|F_1]P[F_1]}{P[G|F_1]P[F_1] + P[G|F_2]P[F_2]} = \frac{0.63}{0.63 + 0.24} = \frac{0.63}{0.87} = 0.72$$

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Example 4: Communication System

Experiment: Noisy channel changes binary signals randomly

Binary bit are transmitted randomly: Bit 0 transmitted with probability 0.9

Bit 1 transmitted with probability 0.1

After bit goes through channel: Output is correct bit with probability 0.95

Output is incorrect bit with probability 0.05

Question: If bit 0 was received, what is probability that a 0 was the original bit?

Answer:

Event a Bit 0 transmitted. $P[a] = 0.9$

Event b Bit 1 transmitted. $P[b] = 0.1$

Event A Bit 0 received. $P[A|a]=0.95$, $P[A|b]=0.05$

Event B Bit 1 received. $P[B|a]=0.05$, $P[B|b]=0.95$

Determine $P[a|A]$

$$P[a|A] = \frac{P[a \cap A]}{P[A]} = \frac{P[A|a]P[a]}{P[A|a]P[a] + P[A|b]P[b]} = \frac{0.9 * 0.95}{0.9 * 0.95 + 0.1 * 0.05} = 0.994$$

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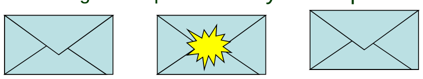
Example 5: Monty Hall Game Show

We have a prize in one of 3 envelopes

- You pick one, sealed.
- I know which envelope has the prize. I open one of two remaining envelopes, and show you that it is empty
- I give you choice of swapping your envelope for the remaining envelope. Should you swap?



- If we stick with original choice we win with probability 1/3.
- If we switch:
  - We loose iff we pick the right envelope initially, hence, we win with probability (1-1/3)=2/3 !

Example 6

3 factories making a product (e.g., 1 kΩ resistors). 70% of product from factory 1 meets spec, 80% from factory 2, 85% from factory 3.

Factory 1 produces 40% of total product, Factory 2 produces 30%, Factory 3 produces 30%

What is probability that a sample product meets spec?

Event G: Product "good". F_i: product comes from factory i.

$$P[G] = \sum_{i=1}^{3} P[G \mid F_i]P[F_i] = 0.775.$$

1.4 Multiple Events

Independence & Conditional Probability

A. Multiplication Rule

$P[A \cap B] = P[B|A]\, P[A]$

$P[A \cap B \cap C] = P[C|A \cap B]\, P[B|A]\, P[A]$

Proof:

$$\begin{aligned} P[A \cap B] &= P[B|A]P[A] \\ P[A \cap B \cap C] &= P[(A \cap B) \cap C)] \\ &= P[C|A \cap B]P[A \cap B] \\ &= P[C|A \cap B]P[B|A]P[A] \end{aligned}$$

B. Conditioning can affect independence

A. Conditional Independence

Given C, *A* and *B* are said to be conditionally independent if:

$P[AB|C] = P[A|C]\, P[B|C]$

Two events that are not independent may become independent conditioned on a third event being true.

Two independent events may become dependent when a third event is observed as true.

C. Independence of multiple events

Events *A*, *B* , and *C* are mutually independent if and only if any pair of these events are independent, and:

$P[A \cap B \cap C] = P[A]\, P[B]\, P[C]$

$A_1, A_2, \ldots, A_n$ are mutually independent if and only if every set of *n-1 events taken from* $A_1, A_2, \ldots, A_n$ are independent and

$P[A_1 \cap A_2 \cap \ldots \cap A_n] = P[A_1]\, P[A_2] \ldots P[A_n]$

Sequential Experiments

Experiment is sequence of sub-experiments

- Select factory, then generate product
- Put bits in channel, let channel corrupt each bit

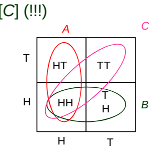
Can generate outcomes sequentially, using a tree

- Simplifies greatly when sub-experiments have some form of independence of events

Example 1 : Independence vs pairwise independence

Two independent samples of a fair coin

- S = {HH, HT, TH, TT}, each atom has probability ¼
- Event A = outcome of first sample is H → $P[A]$ = ½
- Event B = outcome of second sample is H → $P[B]$ = ½
- Event C = outcome of first and second samples are equal → $P[C]$ = ½
- $P[A \cap B] = P[\{HH\}] = ¼ = P[A]\,P[B]$
- $P[A \cap C] = P[\{HH\}] = ¼ = P[A]\,P[C]$ pairwise independent
- $P[B \cap C] = P[\{HH\}] = ¼ = P[B]\,P[C]$
- $P[A \cap B \cap C] = P[\{HH\}] = ¼$, not $P[A]\,P[B]\,P[C]$ (!!!)

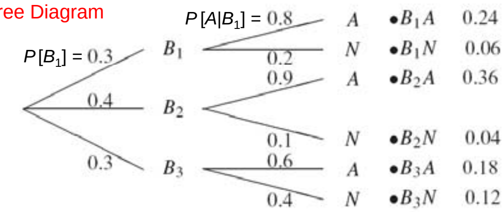


Example 2 : Independence vs pairwise independence

3 machines B_i making resistors

- Machines get selected as first step
- Selected machine generates resistor, Acceptable or Not, as a subexperiment

Tree Diagram



Example 1.24, Fig. 1.2 (p. 25) in book