Lecture 4 - Outline

1. Counting.
2. Reliability theory.

1.5 Principles of Counting

for problems with uniform probability spaces

In many experiments with finite sample spaces \( S \), all outcomes in the sample space \( S \) are "equally likely"

Estimating probabilities requires knowing how many outcomes

\[ P[A] = \frac{\text{outcomes in } A}{\text{total number of outcomes}} \]

This requires counting (combinatorics)

Not central focus of course

1. Number of outcomes of sequential experiments

If subexperiment \( A_1 \) has \( n_1 \) possible outcomes and subexperiment \( A_2 \) has \( n_2 \) possible outcomes then there are \( n_1 n_2 \) possible outcomes when both experiments are performed

For \( r \) subexperiments with \( n_i \) outcomes for subexperiment \( i \), if all choices can be made freely, the total number of outcomes is:

\[ n_1 n_2 \ldots n_r \]

Examples

- 3 pairs of shoes, 2 pairs of pants, 4 shirts: Number of outfits = 3x2x4
- Number of license plates with 3 digits: 3 stages, 10 choices per stage: \( 10^3 = 1000 \)
  Number of license plates with 3 distinct digits: 3 stages: 10 choices at first stage, 9 at 2nd, 8 at 3rd Sampling without replacement...
- Card deck: Number of 5-card poker (7 & up) hands: 5 stages: 8 x 7 x 6 x 5 x 4

2. Sampling

Given a population of \( n \) distinguishable elements \( \{1, \ldots, n\} \), \( k \) elements are sampled.

How many distinguishable samples are possible (if we care about order and if we don’t)?
The sampling may be done with or without replacement.
If sampling is done with replacement, elements of the sample may be repeated.

Example

Population \( n = 26 \)
Sample \( k = 4 \)
A. **Sampling with replacement, order dependent**

\[
\text{\# of possibilities} = n^k
\]

B. **Sampling without replacement, order dependent**

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

C. **Sampling without replacement, order independent**

Number of possible subsets of \(k\) elements out of set of \(n\) is \(\binom{n}{k}\) combinations

D. **Sampling with replacement, order independent**

Also binomial coefficient, but harder formula...

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]

Helps in computing many identities

\[x = 1, y = 1 \Rightarrow 2^n = \sum_{k=0}^{n} \binom{n}{k}\]

**Examples**

- Number of hexadecimal (base-16) numbers with 4 characters: \(16^4\)
  
  Sample (2)(12)(7)(15)

- Number of 8-bit binary symbols with 3 ones:
  
  Sample 01000011

- Birthday Paradox: Consider a group of \(r\) people. The probability they have distinct birthdays is:
  
  \[P[E] = \binom{365}{r} \left(\frac{365-r}{365}\right)^r \left(1 - \frac{r}{365}\right)^{365-r}\]

<table>
<thead>
<tr>
<th>(r)</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.883</td>
</tr>
<tr>
<td>23</td>
<td>0.4927</td>
</tr>
<tr>
<td>60</td>
<td>0.0059</td>
</tr>
</tbody>
</table>

**3. Independent Trials**

Many compound experiments are sequences of independent subexperiments

- Repeated coin tosses
- Multiple bits through communication channel,
- Multiple rolls of 6 sided die

**Objective:** compute probabilities of events

- Total number of heads = \(k\)
- Total number of bit errors = \(k\)
- Sum of numbers = \(k\)

A. **Subexperiments with binary outcomes**

Subexperiment:

- Outcome 1: Success; probability \(p\)
- Outcome 2: Failure; probability \(1-p\)

Experiment: \(n\) subexperiments

- Outcome: a sequence of length \(n\) of failures and successes
- SSFS...F

Event \(E\): \(k\) failures in \(n\) attempts

Probability of \(E = k\) failures in \(n\) attempts

\[P[E] = P[k \text{ failures in } n \text{ attempts}] = \binom{n}{k} (1-p)^k p^{n-k}\]
Communication channel has single bit error rate of $p$, independent across each bit.

Redundant coding of 8 bits into 9 bits allows detection of single bit errors in 8 bit words.

Parity check: makes sure that checksum of all bits is even.

What is probability that a signal with errors is not detected as an error?

What is the probability that the number of errors in 9 bits is even?

Example 1

EC381/MN308 – 2007/2008

Example 2 Quiz 1.9

100 bit packets, single bit prob of error 0.01, can correct 3 bit errors.

$P[k \text{ errors}] = \binom{9}{k} (0.01)^k (0.99)^{9-k}$

$P[\text{uncorrected error}] = 1 - P[0 \text{ errors}] - P[1 \text{ error}] - P[2 \text{ errors}] - P[3 \text{ errors}]$

Example

Calls arriving into a bank’s call center are either concerning Checking accounts (C), Savings accounts (S) or Investment accounts (I).

$P[C] = 0.4, P[S] = 0.4, P[I] = 0.2$.

$S_{c,s,i} = \{\text{Observe c type C, s type S, and i type I calls out of 1000}\}$

$P[S_{c,s,i}] = \binom{1000}{c,s,i} (0.4)^c (0.4)^s (0.2)^i$

Application: Reliability Theory for Interconnected Systems

Assume independent failures of individual subcomponents.

Parallel connection: System works as long as one of subsystems works (redundancy).

Series connection: System works as long as all subsystems work.

Probability of System Failure

For $N$ subsystems in series, each of them has probability $p$ of working correctly.

$E = \text{event system works}; E_i = \text{event system works}$

$P[E] = P[E_1] P[E_2] \ldots P[E_N] = p^N$

For $N$ components in parallel:

$P[E] = P[E_1] P[E_2] \ldots P[E_N] = (1 - p)^N$

$P[E] = 1 - P[E^c] = 1 - (1 - p)^N$
**Example 1.44**

$p = 0.9$. $E = \text{system works}; E_k = \text{subsystem } k \text{ works}$

$E = (E_1 \cap E_2) \cup (E_3 \cap E_4); \quad P[E_1 \cap E_2] = p^2 = P[E_3 \cap E_4]$  

Interpret as parallel connection of 2 subsystems, each with success probability $p^2 \to P[E] = 1 - (1 - p^2)^2$