EC381/MN308
Probability and Some Statistics

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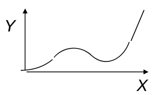
Lecture 6 - Outline

1. Derived Random Variables
2. Conditional PMFs and Expectations
3. Random Vectors

2.3 Derived Random Variables

Given one random variable (RV) X, the function g mapping $R \rightarrow R$ generates the derived RV

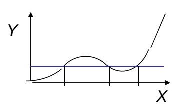
$Y = g(X)$



For outcome s, $X(s)$ is a real number,
$Y(s) = g(X(s))$ is the derived random variable

A. Relation between the PMF of Y and the PMF of X

$$P_Y(y) = \sum_{x: g(x)=y} P_X(x)$$

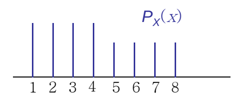


A random variable can be: (1) the observation; (2) a function of the observation; (3) a function of another random variable

Example (Example 2.29 on p. 73)

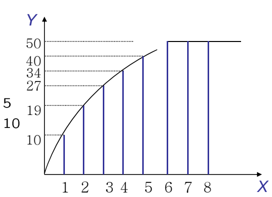
- Prob. of number of pages in fax, X, given by

$$P_X(x) = \begin{cases} 0.15, & 1 \le x \le 4 \\ 0.1 & 5 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$



- Price of pages:

$$Y = g(X) = \begin{cases} 10.5X - 0.5X^2, & 1 \le X \le 5 \\ 50 & 6 \le X \le 10 \end{cases}$$



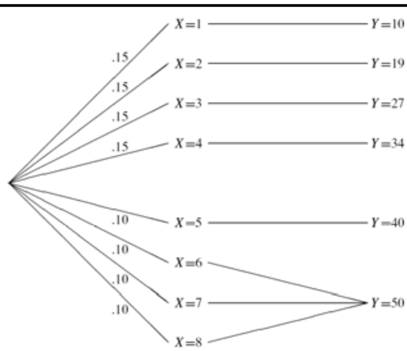
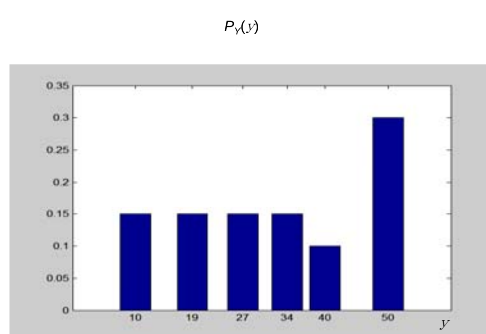


Figure 2.1 (p. 72)
The derived random variable $Y = g(X)$ for this example (Example 2.29).

$P_Y(y)$



Example 2.29 on p. 73

B. Expectation of a Derived RV

$Y = g(X)$ $E[Y] = ?$

$$E[Y] = \boxed{\sum_{y \in S_Y} y P_Y(y)}$$
$$= \sum_{y \in S_Y} y \sum_{x: g(x)=y} P_X(x)$$
$$= \sum_{y \in S_Y} \sum_{x: g(x)=y} g(x) P_X(x) = \boxed{\sum_{x \in S_X} g(x) P_X(x)}$$

Don't need to compute $P_Y(y)$

C. Special Case: Linear Transformation

$$\boxed{Y = aX + b}$$

Expectation

$$E[Y] = \sum_{x \in S_X} g(x) P_X(x)$$
$$= \sum_{x \in S_X} (ax + b) P_X(x)$$
$$= a \sum_{x \in S_X} x P_X(x) + b \sum_{x \in S_X} P_X(x)$$
$$= aE[X] + b$$

Expectation is a linear operation!

Mean	Variance
$E[aX + b] = aE[X] + b$	$\text{Var}[aX + b] = a^2 \text{Var}[X]$

Special Example

$Y = X - m_X$ $m_X = E[X]$

$$E[Y] = E[X] - m_x = m_x - m_x = 0$$

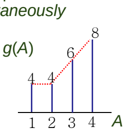
$$Var[Y] = E[Y^2] = E[(X - \mu_X)^2] = Var[X]$$

True for any random variable.

Example (Quiz 2.7 on p. 76)

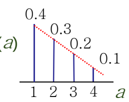
- The number of memory chips M used by PC, depends on # of application programs A *user wants to run simultaneously*
  - 4 chips for 1 or 2 programs
  - 6 chips for 3
  - 8 chips for 4

$M = g(A)$



- *Take* A to be a RV with PMF
  $P_A(a) = 0.1\,(5 - a)$, $a = 1, 2, 3, 4$; and 0 otherwise
- Compute $E[A]$
- Write $M = g(A)$ as a derived variable
- Compute $E[M]$. Is $E[M] = g(E[A])$?

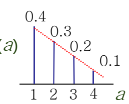
$P_A(a)$



Solution

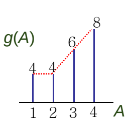
$$E[A] = \sum_{a=1}^{4} a P_A(a) = \sum_{a=1}^{4} 0.1 a (5 - a)$$
$$= 0.1(4 + 6 + 6 + 4) = 2$$

$P_A(a)$



$$M = g(A) = 2A + 2 * I\{A = 1\}$$
$$I\{A = 1\} = \{\text{outcomes such that } A = 1\}$$

$M = g(A)$



$$E[M] = E[g(A)] = 2E[A] + 2 * E[I\{A = 1\}]$$

nonlinear

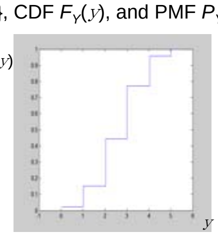
$$= 4 + 2 \sum_{a=1}^{4} I\{a = 1\} P_A(a)$$
$$= 4 + P_A(1) = 4.4 \neq 4 = g(E[A]) = g(2)$$

Generation of a Discrete Random Variable of Prescribed PMF by Transformation of a Uniform Random Variable

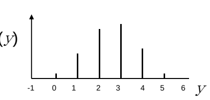
Problem: Generate a discrete RV Y with prescribed range:

$S_Y = \{0, 1, 2, .. k\}$, CDF $F_Y(y)$, and PMF $P_Y(y) = F_Y(y) - F_Y(y-1)$.

Example $F_Y(y)$



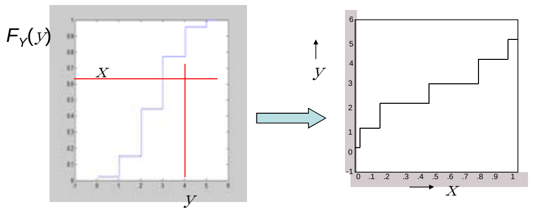
$P_Y(y)$



Solution: Generate a random number X in [0,1] with uniform distribution (MATLAB: rand(1,1)).

Find a transformation $Y = g(X)$ that transforms X into $Y = \{0, 1, 2, .., k\}$, such that $F_Y(y-1) < x \leq F_Y(y)$

Transformation: Lookup table for determining the y corresponding to x



2.4 Conditional PMF

The conditional PMF of X given an event B is

$P_{X|B}(x) = P[X = x \mid B]$

$A = \{X = x\}$ is collection of outcomes

Since $P[A|B] = P[A \cap B] / P[B]$

$$P_{X|B}(x) = P[\{X = x\}|B] = \frac{P[\{X = x\} \cap B]}{P[B]} = \frac{P[X = x, B]}{P[B]}$$

Special case: $B \subset S_X$

Event B = outcomes of X with values in B

A value $x \in S_X$ is either in B or not $x \in B \rightarrow \{X = x\} \cap B = \{X = x\}$

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases}$$

$P_{X|B}(x)$ is a probability mass function with usual properties:

$P_{X|B}(x) \geq 0$

$\sum_{x \in S_X} P_{X|B}(x) = 1$

$\sum_{x \in B} P_{X|B}(x) = 1 \quad \text{if } B \subset S_X$

$\text{if } B \subset S_X, \text{ and } C \subset B, \; P[C|B] = \sum_{x \in C} P_{X|B}(x)$

Conditional Expectation

Conditional PMF $P_{X|B}(x) \equiv P[X = x|B]$

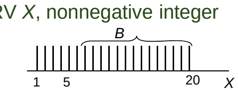
$E[X|B]$ = conditional expectation of RV X given that outcomes are in event B.

Compute using the conditional PMF

$$E[X|B] = \sum_{x \in S_x} x P_{X|B}(x)$$

Example

- Arrival time of a message is a RV X, nonnegative integer valued, with uniform PMF
  - $P_X(x) = 1/20, \quad x = 1, \ldots, 20$
  - $P_X(x) = 0 \quad$ otherwise



- Consider the event $B = \{X > 6\}$ (message does not arrive by 6-th time slot)
  - Compute $P_{X|B}(x|B)$

$$P_{X|B}(x|B) = \begin{cases} \frac{P_X(x)}{P(B)} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1/20}{14/20} = \frac{1}{14} & 6 < x \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

$E[X \mid X > 6] = (20+7)/2 = 13.5$

Conditional Expectations of Derived RVs

$Y = g(X)$

$$E[Y|B] = E[g(X)|B] = \sum_{x \in S_X} g(x) P_{X|B}(x)$$

Example (cont.):

Find variance of X given $X > 6$

$E[X|X > 6] = 13.5$

$$E[X^2|X > 6] = \sum_{x=7}^{20} x^2 \frac{1}{14} = 198.5$$

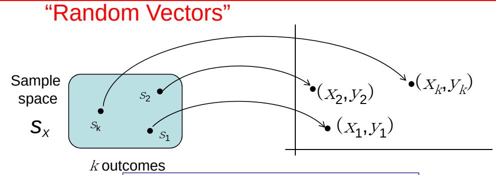
$$\text{Var}(X|X > 6) = E[X^2|X > 6] - (E[X|X > 6])^2 = 16.25$$

$$\sigma_{X|X>6} = \sqrt{16.25} \approx 4.03$$

or we can also use the formula for the variance of a uniform in {7, …, 20}

2.5 Multiple Discrete Random Variables

"Random Vectors"



Two Random Variables X, Y on same probability space

Joint PMF

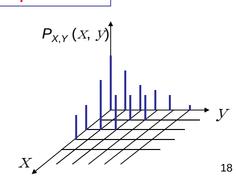
– $P_{X,Y}(x, y) = P[X = x \text{ and } Y = y]$

Properties

$\sum_x \sum_y P_{X,Y}(x, y) = 1$

$P_X(x) = \sum_y P_{X,Y}(x, y)$ Marginal PMF

$P_Y(y) = \sum_x P_{X,Y}(x, y)$ Marginal PMF



Conditional Probability

- Definition.
  - $P_{X|Y}(x \mid y) = P[X = x \mid Y = y] = P_{X,Y}(x, y) / P_Y(y)$
  - Interpretation: Observe outcomes associated with event $\{Y = y\}$ and compute conditional probability mass function of X
- Properties. It is a PMF on X:

$P_{X|Y}(x \mid y)$ in [0,1]

$\sum_x P_{X|Y}(x \mid y) = 1$

Independence

Two discrete random variables X, Y are independent if and only if the pairs of events $\{X = x\}$, $\{Y = y\}$ are independent for all $x \in S_X$, $y \in S_Y$

$P[X = x, Y = y] = P[X = x]\, P[Y = y]$, i.e., $P_{X,Y}(x, y) = P_X(x)\, P_Y(y)$

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
2		1/20	3/20	1/20
1		1/20		

Independent?