2.3 Derived Random Variables

Given one random variable (RV) \( X \), the function \( g \) mapping \( R \rightarrow R \) generates the derived RV \( Y = g(X) \).

For outcome \( s \), \( X(s) \) is a real number, \( Y(s) = g(X(s)) \) is the derived random variable.

A. Relation between the PMF of \( Y \) and the PMF of \( X \):

\[
P_Y(y) = \sum_{x : g(x) = y} P_X(x)
\]

A random variable can be: (1) the observation; (2) a function of the observation; (3) a function of another random variable.

Example (Example 2.29 on p. 73)

- Prob. of number of pages in fax, \( X \), given by:

\[
P_X(x) = \begin{cases} 
0.15, & 1 \leq x \leq 4 \\
0.1, & 5 \leq x \leq 8 \\
0, & \text{otherwise}
\end{cases}
\]

- Price of pages:

\[
y = g(x) = \begin{cases} 
10.5x - 0.5x^2, & 1 \leq x \leq 5 \\
50, & 6 \leq x \leq 10
\end{cases}
\]
B. Expectation of a Derived RV

\[ Y = g(X) \quad \text{or} \quad E[Y] = ? \]

\[ E[Y] = \sum_{y \in S_Y} y P_Y(y) \]
\[ = \sum_{y \in S_Y} y \sum_{x \in S_X} P_X(x) \]
\[ = \sum_{y \in S_Y} \sum_{x \in S_X} g(x) P_X(x) \]
\[ = \sum_{x \in S_X} g(x) P_X(x) \]

Don’t need to compute \( P_Y(y) \)

C. Special Case: Linear Transformation

\[ Y = aX + b \]

\[ E[Y] = \sum_{x \in S_X} g(x) P_X(x) \]
\[ = \sum_{x \in S_X} (ax + b) P_X(x) \]
\[ = a \sum_{x \in S_X} x P_X(x) + b \sum_{x \in S_X} P_X(x) \]
\[ = aE[X] + b \]

Expectation is a linear operation!

Example (Quiz 2.7 on p. 76)

- The number of memory chips \( M \) used by PC, depends on \# of application programs \( A \) user wants to run simultaneously:
  - 4 chips for 1 or 2 programs
  - 6 chips for 3
  - 8 chips for 4

- Take \( A \) to be a RV with PMF \( P_A(a) = 0.1(5-a) \), \( a = 1, 2, 3, 4 \); and 0 otherwise
- Compute \( E[A] \)
- Write \( M = g(A) \) as a derived variable
- Compute \( E[M] \). Is \( E[M] = g(E[A]) \)?

Generation of a Discrete Random Variable of Prescribed PMF by Transformation of a Uniform Random Variable

Problem: Generate a discrete RV \( Y \) with prescribed range: \( S_Y = \{0, 1, 2, \ldots, \} \), CDF \( F_Y(y) \), and PMF \( P_Y(y) = F_Y(y) - F_Y(y-1) \).

Example \( F_Y(y) \)

<table>
<thead>
<tr>
<th>( y )</th>
<th>0</th>
<th>0.1</th>
<th>0.3</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_Y(y) )</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
<td>0.4</td>
</tr>
</tbody>
</table>

Solution

\[ E[A] = \sum_{a=1}^{4} a P_A(a) = \sum_{a=1}^{4} 0.1(5-a) \]
\[ = 0.1(4+6+6+4) = 2 \]

\[ M = g(A) = 2A + 2 \cdot I(A = 1) \]
\[ I(A = 1) = \{ \text{outcomes such that } A = 1 \} \]

\[ E[M] = E[g(A)] = 2E[A] + 2 \cdot E[I(A = 1)] \]
\[ = 4 + 2 \sum_{a=1}^{4} I(a = 1) P_A(a) \]
\[ = 4 + P_A(1) = 4.4 \neq 4 = g(E[A]) = g(2) \]
Solution: Generate a random number $X$ in $[0,1]$ with uniform distribution (MATLAB: `rand(1,1)). Find a transformation $Y = g(X)$ that transforms $X$ into $Y = \{0, 1, 2, \ldots\}$ such that $F_Y(y-1) < x \leq F_Y(y)$.

Transformation: Lookup table for determining the $y$ corresponding to $x$.

Graph of $F_X(x)$ and $F_Y(y)$.

**2.4 Conditional PMF**

The conditional PMF of $X$ given an event $B$ is

$$P_{X|B}(x) = P[X = x | B]$$

$A \times \{x \in \mathbb{S}\}$ is collection of outcomes

Since $P[A|B] = P[A \cap B] / P[B]$

$$P_{X|B}(x) = \frac{P[X = x \cap B]}{P[B]}$$

Special case: $B = \mathbb{S}_X$.

Event $B$ = outcomes of $X$ with values in $B$.

A value $x \in \mathbb{S}_X$ is either in $B$ or not

$x \in B \rightarrow \{X = x\} \cap B \neq \emptyset$

$$P_{X|B}(x) = \begin{cases} \frac{P[X \in B]}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases}$$

**Example**

- Arrival time of a message is a RV $X$, nonnegative integer valued, with uniform PMF
  - $P_X(x) = \frac{1}{20}, x = 1, \ldots, 20$
  - $P_X(x) = 0$ otherwise

- Consider the event $B = \{X > 6\}$ (message does not arrive by 6-th time slot)
  - Compute $P_{X|B}(x)$

$$P_{X|B}(x) = \begin{cases} \frac{P[X = x]}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases}$$

$$P_{X|B}(x) = \begin{cases} \frac{1}{20} & 6 < x \leq 20 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X | X > 6] = \frac{(20+7)2}{11} = 13.5$$

**Conditional Expectations of Derived RVs**

$$Y = g(X)$$

$$E[Y|B] = E[g(X)|B] = \sum_{x \in \mathbb{S}_X} g(x) P_{X|B}(x)$$

**Example (cont):**

Find variance of $X$ given $X > 6$

$$E[X|X > 6] = \sum_{x=7}^{20} x^2 \frac{1}{14} = 198.5$$

$$\text{Var}(X|X > 6) = E[X^2|X > 6] - (E[X|X > 6])^2 = 16.25$$

$$\sigma_{X|X > 6} = \sqrt{16.25} \approx 4.03$$

or we can also use the formula for the variance of a uniform in $[7, \ldots, 20]$.

**2.5 Multiple Discrete Random Variables**

"Random Vectors"

Sample space $\mathbb{S}_X$.

Two Random Variables $X$, $Y$ on same probability space

Joint PMF

$$P_{X,Y}(x, y) = P[X = x \text{ and } Y = y]$$

Properties

$$\sum_x \sum_y P_{X,Y}(x,y) = 1$$

$$P_X(x) = \sum_y P_{X,Y}(x,y) \text{ Marginal PMF}$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y) \text{ Marginal PMF}$$

Graph of $P_{X,Y}(x,y)$.
**Conditional Probability**

- **Definition.**
  - \( P_{X|Y}(x|y) = P[X = x | Y = y] = P_{X,Y}(x,y) / P_Y(y) \)
  - Interpretation: Observe outcomes associated with event \( \{Y = y\} \) and compute conditional probability mass function of \( X \)

- **Properties.** It is a PMF on \( X \):
  - \( P_{X|Y}(x|y) \) in \([0,1]\)
  - \( \sum_x P_{X|Y}(x|y) = 1 \)