EC381/MN308
Probability and Some Statistics

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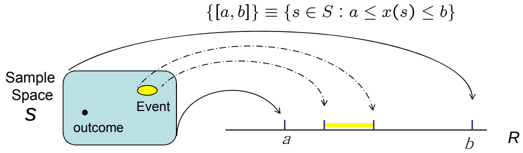
Lecture 7 - Outline

1. Continuous Random Variables

Chapter 3 Continuous Random Variables

Continuous Random Variables = RV whose range is a continuous subset of the real number, i.e., takes an uncountably infinite number of possible values

$$\{(a,b)\} \equiv \{s \in S : a < x(s) < b\}$$
$$\{[a,b)\} \equiv \{s \in S : a \le x(s) < b\}$$
$$\{(a,b]\} \equiv \{s \in S : a < x(s) \le b\}$$
$$\{[a,b]\} \equiv \{s \in S : a \le x(s) \le b\}$$



*Continuous Random Variable **X***

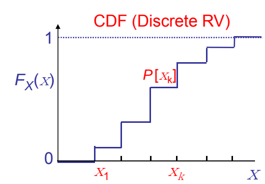
Key issue: Often cannot associate a nonzero probability with any individual outcome. Cannot enumerate "experiment" outcomes. Can only define probabilities of events!

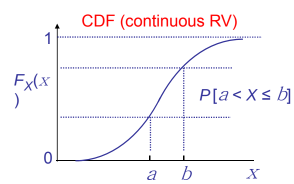
Will focus on events representing outcomes where random variables take values in *intervals*

3.1 Cumulative Distribution Function (CDF)

The *cumulative (probability) distribution function* of a random variable X (continuous or discrete) is the total probability mass from $-\infty$ up to (and including) the point x:

$$F_X(x) = P[X \le x], \quad \text{or} \quad F_X(x) = P[X \in (-\infty, x]]$$





$P[a < X \le b] = F_X(b) - F_X(a)$

Properties of the CDF

- $F_X(x)$ is a non-negative real-valued function defined for all real number values of its argument x $(-\infty < x < \infty)$
- $F_X(x) \in [0,1]$, $\quad F_X(-\infty) = 0$ and $F_X(+\infty) = 1$
- $F_X(x)$ is a monotonic nondecreasing function of x, i.e.,

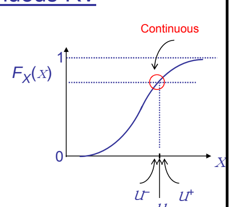
  if $a < b$, then $F_X(a) \le F_X(b)$

Properties of the CDF for continuous RV

- $F_X(x)$ is a continuous function of x,

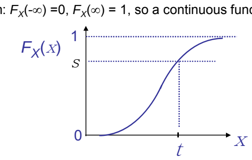
  i.e., $F_X(u) = F_X(u^+) = F_X(u^-)$
- $P[X = u] = 0$. Every single-value event $\{X = u\}$ has zero probability. Such an event is physically unobservable since it requires an instrument with infinite precision.
- $P[X < x] = P[X \le x] = F_X(x)$
- Nonzero probabilities are assigned to intervals of the line.

  For $a < b$, $P[a < X \le b] = F_X(b) - F_X(a)$

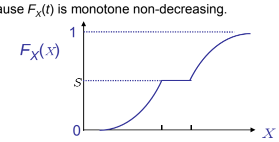


Other properties of the continuous CDF

- If s is any number in the range $0 < s < 1$, then there must be at least one number t such that $F_X(t) = s$
  - Intermediate value theorem: $F_X(-\infty) = 0$, $F_X(\infty) = 1$, so a continuous function takes all values between 0 and 1



- Can there be many t such that $F_X(t) = s$?
  - This could happen: there could be an interval of t's such that $F_X(t) = s$
  - Must be a single interval, because $F_X(t)$ is monotone non-decreasing.

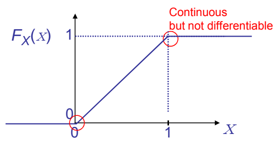


Example

Choose a random number between 0 and 1, i.e., $S = \{x : 0 < x < 1\}$

Intuitively, the meaning of random in this instance is that we do not favor any one number over others in the interval (0,1)

One way of expressing the innate randomness of the choice is as follows: Given any subinterval of (0, 1), the probability that the chosen number lies in that subinterval is equal to the length of that interval



P.S. This is an example of a continuous, but not everywhere differentiable, CDF

Advanced topic:

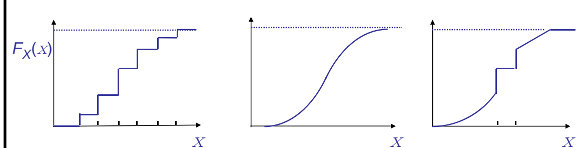
More general definition of continuous RVs

- A continuous random variable X is one whose CDF, $F_X(x)$, is
  1. continuous at all x, $-\infty < x < \infty$
  2. differentiable at all x except possibly at a *countable* set of points $x_1 < x_2 < \ldots x_n < \ldots$
- More precisely, any finite-length interval contains at most a finite number of points where $F_X(x)$ is not differentiable. The CDF of a discrete random variable also satisfies this condition.

Note: The book does not stress that there can be points where $F_X(u)$ is continuous but not differentiable

Mixed (Continuous/Discrete) RVs

Mixed RVs have piecewise differentiable CDFs with positive slopes and jump discontinuities

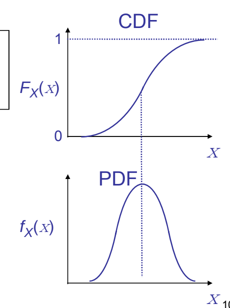


Discrete RV Continuous RV Mixed RV

3.2 Probability Density Function (PDF)

For a continuous RV X with a CDF $F_X(x)$ that is differentiable almost everywhere, the probability density function (PDF) is the derivative of the CDF :

$$f_X(x) = \begin{cases} \frac{d}{dx}F_X(x) & \text{if diff. at } x \\ \text{number } \geq 0 & \text{else} \end{cases}$$



The PDF of a continuous RV is *not* a probability and may take values greater than one. It is a probability density. However, the integral of a PDF over a region of x is a probability.

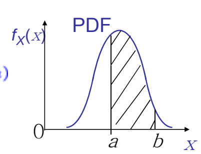
Properties of the PDF

- $f_X(x) \geq 0$, i.e., PDF is a non-negative function
- $P[a < X < b] = P[a \leq X < b] = P[a \leq X \leq b]$

$$= \int_a^b f_X(x)dx$$ = area under the curve $f_X(x)$ from a to b.

Proof:

$$\int_a^b f_X(x)dx = \int_a^b \frac{dF_X(x)}{dx}dx = F_X(b) - F_X(a)$$
$$= P[a < x \leq b]$$



- $\int_{-\infty}^{\infty} f_X(x)dx = 1$ i.e., the PDF has unit area
- $f_X(+\infty) = 0$; $f_X(-\infty) = 0$, i.e., as the argument x tends to $\pm\infty$, the PDF curve must decay away to 0 (or else the area under it would not be finite). The slope of the CDF $F_X(x)$ goes to 0 as $|x| \rightarrow \infty$

PDF versus PMF

- PMF of a Discrete RV defines a set of point masses on the axis: Total mass = 1. PMF $P_X(x)$ = mass at x = probability that x occurs.
- PDF of a Continuous RV defines a spread of the total probability mass of 1 along the axis. There is no probability mass at any point.
- PDF of a continuous RV is not a probability. It provides the density of the mass at each point
  - The PDF is measured in units of probability mass/length
  - The PDF is analogous to mass or charge density, etc.
- If $f_X(a)$ is positive at the point a and δ is the length of an interval, then

$$P[a \leq X \leq a + \delta] \approx f_X(a) \cdot \delta$$

The approximation becomes better as δ becomes smaller.

Example 1

A continuous random variable X has PDF $f_X(x) = 0.75(1 - x^2)$ for $-1 \le x \le 1$, and 0 otherwise. Compute $P[0.25 \le X \le 1.25]$:

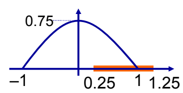
$$P(0.25 \le X \le 1.25) = \int_{0.25}^{1.25} 0.75(1-x^2)dx$$
$$= 0.75(x - \frac{x^3}{3})|_{0.25}^{1.25}$$
$$= 17/64$$

Incorrect

Most problems on continuous random variables are easier to visualize with a diagram, which helps in figuring out the limits and avoiding errors.

Correct answer

$$P(0.25 \le X \le 1.25) = \int_{0.25}^{1} 0.75(1-x^2)dx$$
$$= 0.75(u - \frac{x^3}{3})|_{0.25}^{1}$$
$$= 81/256$$



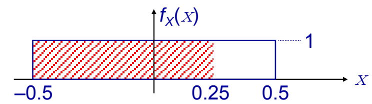
Example 2

A continuous random variable X has PDF $f_X(x) = 1$ for $-0.5 \le x \le 0.5$, and 0 otherwise:

$$P(X < 0.25) = \int_{0}^{0.25} f_x(x)dx$$
$$= \int_{0}^{0.25} dx = 0.25$$

Incorrect

Correct answer



$P[X \le 0.25]$ = area under PDF to left of 0.25 = shaded area = 0.75

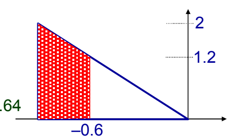
Example 3 Finding the CDF from the PDF

$f_X(x) = -2x$, for $-1 \le x \le 0$,
= 0, otherwise

$F_X(-0.6) = P[X \le -0.6] = P[-\infty < X \le -0.6]$

= area under PDF from $-\infty$ to -0.6

$= 1 - (1/2) \cdot 0.6 \cdot 1.2 = 1 - (0.6)^2 = 0.64$

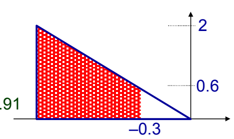


$F_X(-0.3) = P[X \le -0.3] = P[-\infty < X \le -0.3]$

= area under PDF from $-\infty$ to -0.3

$= 1 - (1/2) \cdot 0.3 \cdot 0.6 = 1 - (0.3)^2 = 0.91$

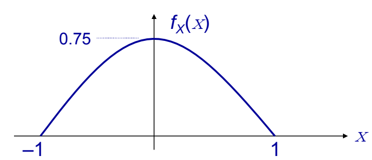
(increases as it should)



Example 4

$f_X(x) = 0.75(1 - x^2)$ for $-1 \le x \le 1$,
= 0, otherwise

Find $F_X(0)$:



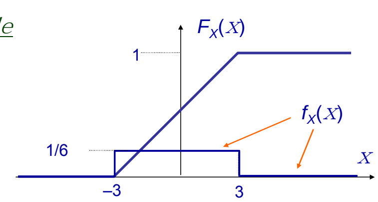
$P[X \le 0]$ = area under PDF curve to the left of 0
= 1/2 by symmetry!

Advanced topic: PDF for non-differentiable CDF

$$f_X(x) = \begin{cases} \frac{d}{dx}F_X(x) & \text{if } F_X(x) \text{ is differentiable at } x \\ \text{any number } \ge 0 & \text{if CDF is not diff. at } x \end{cases}$$

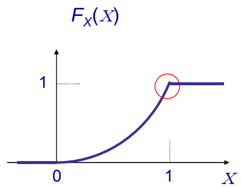
- The derivative of the CDF of a continuous random variable X exists for almost all real numbers x
- We are allowed to set the value of $f_X(x)$ to any nonnegative number only at those few isolated points where the CDF is not differentiable
- Furthermore, the arbitrarily chosen value assigned to the pdf at these isolated points makes no difference whatsoever in any probability calculations
  - The probability that this number occurs is 0!

Example



- The derivative of the CDF is undefined at x = +3 or - 3
  - Could choose value as right derivative or left derivative, but doesn't matter as long as value is nonnegative

Example of convention



$$F_X(x) = \begin{cases} 0 & x \le 0 \\ x^2 & 0 < x < 1 \\ 1 & x \ge 1 \end{cases}$$

CDF Not differentiable at $X = 1$.

$f_X(X)$ has value $2X$ for $0 \le X < 1$ *BUT* $f_X(1)$ = ?

Two convenient choices:

- $f_X(X) = 2X$ for $0 \le X \le 1$, and 0 elsewhere
  i.e., X takes on values in [0, 1]
- $f_X(X) = 2X$ for $0 < X < 1$, and 0 elsewhere
  i.e., X takes on values in (0, 1)

3.3 Expectation

Definition:
The expected value (average) of a random variable X is

$$E[X] = \sum_{x \in S_X} x p_X(x)$$

Discrete

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

Continuous

Significance:

- If we repeat an experiment N times, add up all observed values of X, and divide by N, the result will be pretty close to $E[X]$
- Center of probability mass, center of gravity, …

Expectation of a Function of a Random Variable

$g(X)$ is a function of a continuous random variable X

$g(X)$ is not necessarily a continuous function

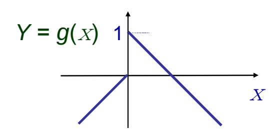
By analogy with discrete random variables:

$$E[Y] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Example

$Y = X$, for $X \le 0$,

$Y = 1 - X$, for $X > 0$



$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

$$= \int_{-\infty}^{0} x f_x(x) dx + \int_{0}^{\infty} (1 - x) f_x(x) dx$$

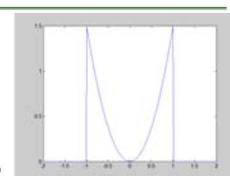
Moments and Central Moments

Definitions (same as for discrete random variables)

$\mu_X \equiv E[X]$	= mean (first moment) of X
$E[X^n]$	= n-th moment of X
$E[X - a]$	= (first) moment of X about a
$E[(X - a)^n]$	= n-th moment of X about a
$E[(X - \mu_X)^n]$	= n-th central moment of X
$E[X - \mu_X]$	= first central moment of X
$E[(X - \mu_X)^2]$	= second central moment of X = Var[X] = Variance
Var[X]	= $E[X^2] - \mu_X^2$ (similar to moment of inertia around mean)
$\sigma_X = \{\text{Var}[X]\}^{1/2}$	= Standard deviation = spread around mean (similar to radius of gyration)

Example Quiz 3.3

RV Y, with PDF :



```
x = linspace(-2,2,1000);
mask = find((x >= -1)&(x <= 1));
y = zeros(size(x));
y(mask) = 3*x(mask).*x(mask)/2;
plot(x,y)
```

$$f_Y(y) = \begin{cases} 3y^2/2 & -1 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Y] = \int_{-1}^{1} y \frac{3y^2}{2} dy = 0 \text{ (odd fcn)}$$

$$\text{Var}[Y] = \int_{-1}^{1} (y - 0)^2 \frac{3y^2}{2} dy$$

$$= \frac{3y^5}{10}\Big|_{-1}^{1} = 3/5$$

$$\sigma_Y = \sqrt{\text{Var}[Y]} = \sqrt{3/5}$$

Expectations of Linear Functions of a RV (same as for discrete case)

$$Y = aX + b$$

Mean $E[aX + b] = aE[X] + b$

The mean is multiplied by the same factor a and shifted by the same factor b.
The expectation is a linear operation, i.e., the expectation of a weighted sum is the weighted sum of the expectations

Variance $\mathrm{Var}[aX + b] = a^2\,\mathrm{Var}[X]$

The variance is multiplied by the square of a and is insensitive to the shift factor b.

Proof

$\mathrm{Var}[aX] = E[(aX)^2] - (E[aX])^2 = a^2 E[X^2] - (a\mu_X)^2$

$= a^2(E[X^2] - (\mu_X)^2) = a^2\,\mathrm{Var}[X]$

Expectations cannot always be defined:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{0} x f_X(x)dx + \int_{0}^{\infty} x f_X(x)dx$$

Since $xf_X(x) \geq 0$ for $X > 0$, but $xf_X(x) \leq 0$ for $X < 0$, $E[X]$ is the *difference* between two positive integrals over $(0, \infty)$ and $(-\infty, 0)$. If both integrals are infinite, $E[X]$ is undefined. This does not mean, however, that the PDF is not useful (random fractals are often characterized by such PDFs).

Example

The Cauchy RV has a PDF $f_X(X) = [\pi(1+X^2)]^{-1}$

This PDF is symmetric about the origin but has long (power-law) tails.

The integral of $X[(\pi(1+X^2)]^{-1}$ is of the form $\infty - \infty$ and $E[X]$ is undefined.

Some RVs have finite means but higher moments are undefined.