SE524/EC524 Optimization Theory and Methods

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Lecture 10: Outline

1. Farkas' lemma.
2. Application: Asset Pricing in an arbitrage-free environment.
3. Cones and extreme rays.
4. Unboundness conditions.
5. Resolution Theorem.

Farkas' lemma

Theorem

(Farkas' lemma) Exactly one of the following two alternatives hold:

- (a) $\exists \mathbf{x} \geq \mathbf{0}$ s.t. $\mathbf{Ax} = \mathbf{b}$.
- (b) $\exists \mathbf{p}$ s.t. $\mathbf{p}'\mathbf{A} \geq \mathbf{0}'$ and $\mathbf{p}'\mathbf{b} < 0$.

Corollary

Assume that any $\mathbf{p}$ satisfying $\mathbf{p}'\mathbf{A}_i \geq 0$, also satisfies $\mathbf{p}'\mathbf{b} \geq 0$. Then b can be written as a nonnegative lin. combination of $\mathbf{A}_1, \ldots, \mathbf{A}_n$.

Theorem

Suppose $\mathbf{Ax} \leq \mathbf{b}$ has at least one feasible solution. Let d scalar. Then the following are equivalent:

- (a) $\forall$ feasible sols. of $\mathbf{Ax} \leq \mathbf{b}$ we have $\mathbf{c}'\mathbf{x} \leq d$.
- (b) $\exists \mathbf{p} \geq \mathbf{0}$ s.t. $\mathbf{p}'\mathbf{A} = \mathbf{c}'$ and $\mathbf{p}'\mathbf{b} \leq d$.

Cones

Definition

A set $C \subset \mathbb{R}^n$ is a **cone** if $\lambda \mathbf{x} \in C \; \forall \lambda \geq 0$ and $\forall \mathbf{x} \in C$.

Definition

Polyhedral cone: $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \geq \mathbf{0}\}$. If $\mathbf{0}$ is an extreme point we have a **pointed polyhedral cone**.

Theorem

Let polyhedral cone $C \subset \mathbb{R}^n$ s.t. $C = \{\mathbf{x} \mid \mathbf{a}_i'\mathbf{x} \geq 0\}$. Then the following are equivalent:

- (a) $\mathbf{0}$ is an extreme point of C.
- (b) C does not contain a line.
- (c) $\exists$ n lin. ind. vectors in $\mathbf{a}_1', \ldots, \mathbf{a}_m'$.

Recession cones and extreme rays

Let nonempty polyhedron $P = \{\mathbf{x} \mid \mathbf{Ax} \geq \mathbf{b}\}$.

Definition

Recession cone at $\mathbf{y} \in P$:

$$\{\mathbf{d} \mid \mathbf{A}(\mathbf{y} + \lambda \mathbf{d}) \geq \mathbf{b}\ \forall \lambda \geq 0\} \Rightarrow \{\mathbf{d} \mid \mathbf{Ad} \geq \mathbf{0}\}$$

d in recession cone are called **rays** of polyhedron.

Extreme rays of polyhedral cone C: $\mathbf{x} \in C$ s.t. $n-1$ lin. ind. constraints are active at $\mathbf{x}$.
Extreme rays of recession cone of P are called **extreme rays of P**.

Unboundness Conditions

Theorem

Consider $\min \mathbf{c}'\mathbf{x}$ over pointed $C = \{\mathbf{x} \mid \mathbf{a}_i'\mathbf{x} \geq 0\}$. Cost $= -\infty$ **iff** $\exists$ extreme ray $\mathbf{d}$ with $\mathbf{c}'\mathbf{d} < 0$.

Theorem

Consider $\min \mathbf{c}'\mathbf{x}$ over $\mathbf{Ax} \geq \mathbf{b}$. Assume at least one extreme point exists. Cost $= -\infty$ **iff** $\exists$ extreme ray $\mathbf{d}$ with $\mathbf{c}'\mathbf{d} < 0$.

Resolution Theorem

Theorem

Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\} \neq \varnothing$ with at least one extreme point. Let $\mathbf{x}^1, \ldots, \mathbf{x}^k$ be the extreme points and $\mathbf{w}^1, \ldots, \mathbf{w}^r$ a complete set of rays. Then

$$Q \triangleq \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}^i + \sum_{j=1}^{r} \theta_j \mathbf{w}^j \mid \lambda_i \geq 0, \theta_j \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\} = P$$

Converse is also true: Every set of the form of Q (finitely generated) is a polyhedron.