SE524/EC524 Optimization Theory and Methods

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Lecture 11: Outline

1. Local sensitivity analysis.
2. Dependence of optimal cost on **b**.
3. Dependence of optimal cost on **c**.

Local Sensitivity

Consider an LP solved to optimality. Let $\mathbf{B}$ be an optimal basis and $\mathbf{x}^*$ the associated optimal solution. Then

$$\mathbf{B}^{-1}\mathbf{b} \geq 0 \qquad \textbf{(feasibility)}$$
$$\bar{\mathbf{c}}' = \mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} \geq \mathbf{0}' \qquad \textbf{(optimality)}$$

How $\mathbf{B}$ and $\mathbf{x}^*$ are affected when:

1. A new variable is added ?
2. A new inequality constraint is added ?
3. A new equality constraint is added ?
4. We change $\mathbf{b}$?
5. Changes in $\mathbf{c}$?
6. Changes in elements of $\mathbf{A}$?

Optimal cost dependence on b

$$P(\mathbf{b}) \triangleq \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$
$$S \triangleq \{\mathbf{b} \mid P(\mathbf{b}) \neq \varnothing\}$$
$$F(\mathbf{b}) \triangleq \min_{\mathbf{x} \in P(\mathbf{b})} \mathbf{c}'\mathbf{x}, \quad \text{for } \mathbf{b} \in S$$

Assume that the dual feasible set $\{\mathbf{p} \mid \mathbf{p}'\mathbf{A} \leq \mathbf{c}'\} \neq \varnothing$. Let $\mathbf{b}^* \in S$ associated with a nondegenerate (primal) optimal solution. Then

$$F(\mathbf{b}) = \mathbf{p}'\mathbf{b} \qquad \text{for } \mathbf{b} \text{ "close" to } \mathbf{b}^*$$

Theorem

$F(\mathbf{b})$ is a convex function of $\mathbf{b}$ on the set S.

Optimal cost dependence on $\mathbf{c}$

$$Q(\mathbf{c}) \triangleq \{\mathbf{p} \mid \mathbf{p}'\mathbf{A} \leq \mathbf{c}\}$$

$$T \triangleq \{\mathbf{c} \mid Q(\mathbf{c}) \neq \varnothing\}$$

$$G(\mathbf{c}) \triangleq \min_{\substack{\mathbf{A}\mathbf{x}=\mathbf{b} \\ \mathbf{x}\geq\mathbf{0}}} \mathbf{c}'\mathbf{x}$$

Theorem

a. T is a convex set.
b. $G(\mathbf{c})$ is a concave function of $\mathbf{c}$ on T.
c. If for some $\mathbf{c}$ the primal has unique optimal, say $\mathbf{x}^*$, then G is linear in the vicinity of $\mathbf{c}$ with gradient $\mathbf{x}^*$.