# SE524/EC524 Optimization Theory and Methods

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## Lecture 14: Outline

1. Dantzig-Wolfe decomposition.
2. Stochastic Programming and Benders Decomposition.

# Dantzig-Wolfe decomposition

Consider the LP ($\mathbf{x}_1 \in \mathbb{R}^{n_1}, \mathbf{x}_2 \in \mathbb{R}^{n_2}$):

$$\begin{array}{lll} \min & \mathbf{c}_1'\mathbf{x}_1 + \mathbf{c}_2'\mathbf{x}_2 & \\ \text{s.t.} & \mathbf{D}_1\mathbf{x}_1 + \mathbf{D}_2\mathbf{x}_2 = \mathbf{b}_0 & (m_0) \\ & \mathbf{F}_1\mathbf{x}_1 = \mathbf{b}_1 & (m_1) \\ & \mathbf{F}_2\mathbf{x}_2 = \mathbf{b}_2 & (m_2) \\ & \mathbf{x}_1, \mathbf{x}_2 \geq \mathbf{0}. & \end{array}$$





# Reformulation

$$\mathscr{P}_i = \{\mathbf{x}_i \geq \mathbf{0} \mid \mathbf{F}_i\mathbf{x}_i = \mathbf{b}_i\} \neq \varnothing$$

By **resolution theorem**:

$$\mathbf{x}_i = \sum_{j \in J_i} \lambda_i^j \mathbf{x}_i^j + \sum_{k \in K_i} \theta_i^k \mathbf{w}_i^k, \qquad \sum_{j \in J_i} \lambda_i^j = 1.$$

Substitute to obtain **Master Problem**

$$\begin{array}{l} \min \sum_{j \in J_1} \lambda_1^j \mathbf{c}_1'\mathbf{x}_1^j + \sum_{k \in K_1} \theta_1^k \mathbf{c}_1'\mathbf{w}_1^k + \sum_{j \in J_2} \lambda_2^j \mathbf{c}_2'\mathbf{x}_2^j + \sum_{k \in K_2} \theta_2^k \mathbf{c}_2'\mathbf{w}_2^k \\ \text{s.t.} \sum_{j \in J_1} \lambda_1^j \begin{bmatrix} \mathbf{D}_1\mathbf{x}_1^j \\ 1 \\ 0 \end{bmatrix} + \sum_{j \in J_2} \lambda_2^j \begin{bmatrix} \mathbf{D}_2\mathbf{x}_2^j \\ 0 \\ 1 \end{bmatrix} + \sum_{k \in K_1} \theta_1^k \begin{bmatrix} \mathbf{D}_1\mathbf{w}_1^k \\ 0 \\ 0 \end{bmatrix} + \sum_{k \in K_2} \theta_2^k \begin{bmatrix} \mathbf{D}_2\mathbf{w}_2^k \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ 1 \\ 1 \end{bmatrix} \\ \quad \lambda_i^j \geq 0, \ \theta_i^k \geq 0, \qquad \forall\ i, j, k. \end{array}$$

Only $m_0 + 2$ constraints, Huge # of variables.

# Subproblems

Use Revised simplex. Let $\mathbf{B}$ basis, $\mathbf{p}' = [\mathbf{q}' r_1 r_2]$ dual vector.
Check all reduced costs of nonbasic variables ? **Hopeless !**

$$\text{Reduced Cost of } \lambda_1^j = (\mathbf{c}_1' - \mathbf{q}'\mathbf{D}_1)\mathbf{x}_1^j - r_1$$
$$\text{Reduced Cost of } \theta_1^k = (\mathbf{c}_1' - \mathbf{q}'\mathbf{D}_1)\mathbf{w}_1^k$$

**Subproblem I** min $(\mathbf{c}_1' - \mathbf{q}'\mathbf{D}_1)\mathbf{x}_1$
s.t. $\mathbf{x}_1 \in \mathscr{P}_1$

**Subproblem II** min $(\mathbf{c}_2' - \mathbf{q}'\mathbf{D}_2)\mathbf{x}_2$
s.t. $\mathbf{x}_2 \in \mathscr{P}_2$





# Possible Outcomes

1. Cost$=-\infty$. Generate column.
2. Cost Finite $< r_1$. Generate column.
3. Cost Finite $\geq r_1$.

Iterate until optimality.

**Remarks**

- Do not need standard form.
- Can have more than two subproblems.

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## Initialization

- Need a BFS for the master problem. Apply Phase I to $\mathscr{P}_1, \mathscr{P}_2$ and obtain extreme points $\mathbf{x}_1^1, \mathbf{x}_2^1$. Assume $\mathbf{D}_1\mathbf{x}_1^1 + \mathbf{D}_2\mathbf{x}_2^1 \leq \mathbf{b}_0$ (may have to multiply with $-1$).
- Introduce auxiliary variable $\mathbf{y} \in \mathbb{R}^{m_0}$ and consider:

$$
\begin{array}{ll}
\min & \sum_{t=1}^{m_0} y_t \\
\text{s.t.} & \sum_{i=1,2}\left(\sum_{j\in J_i} \lambda_i^j \mathbf{D}_i\mathbf{x}_i^j + \sum_{k\in K_i} \theta_i^k \mathbf{D}_i\mathbf{w}_i^k\right) + \mathbf{y} = \mathbf{b}_0 \\
& \sum_{j\in J_1} \lambda_1^j = 1 \\
& \sum_{j\in J_2} \lambda_2^j = 1 \\
& \lambda_i^j \geq 0,\ \theta_i^k \geq 0, y_t \geq 0 \qquad \forall\ i,j,k,t.
\end{array}
$$

- Initial BFS: $\lambda_1^1 = \lambda_2^1 = 1,\ \lambda_i^j = 0\ j \neq 1, i = 1,2,$ $\theta_i^k = 0\ \forall k, i = 1,2,\ y = \mathbf{b}_0 - \mathbf{D}_1\mathbf{x}_1^1 - \mathbf{D}_2\mathbf{x}_2^1$.
- Use D-W decomposition: if cost $> 0$ then infeasible; else we have BFS for master problem.



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## Bounds on optimal cost

- $z^*$: optimal cost.
- $z$: cost of feasible solution at some intermediate level.
- $r_i$: dual variable associated with convexity constraint of $i$th subproblem.
- $z_i$: optimal cost of $i$th subproblem.

**Theorem**

$z + \sum_i (z_i - r_i) \leq z^* \leq z.$

## Stochastic Programming

Suppose we want to make decisions in two stages.

- First choose $\mathbf{x}$.
- Then scenario $\omega$, out of $K$, reveals itself. $\mathbf{P}[\omega] = a_\omega$.
- Then want to choose $\mathbf{y}_\omega$.

**First stage constraints**: $\mathbf{A}\mathbf{x} = \mathbf{b}$
$\mathbf{x} \geq \mathbf{0}$

**Both stage constraints**: $\mathbf{B}_\omega \mathbf{x} + \mathbf{D}\mathbf{y}_\omega = \mathbf{d}_\omega$
$\mathbf{y}_\omega \geq \mathbf{0}$

**Objective**: minimize $\mathbf{E}[\text{Cost}] = \min \mathbf{c}'\mathbf{x} + \sum_{i=1}^{K} a_i \mathbf{f}'\mathbf{y}_i$





## Formulation: Master Problem

$$
\begin{aligned}
\min \quad & \mathbf{c}'\mathbf{x} + \sum_{i=1}^{K} a_i \mathbf{f}'\mathbf{y}_i \\
& \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{B}_1\mathbf{x} + \mathbf{D}\mathbf{y}_1 = \mathbf{d}_1 \\
& \quad \vdots \\
& \mathbf{B}_k\mathbf{x} + \mathbf{D}\mathbf{y}_k = \mathbf{d}_k \\
& \mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_k \geq 0
\end{aligned}
$$

Lots of decision variables ...