SE524/EC524 Optimization Theory and Methods

Yannis Paschalidis
yannisp@bu.edu, http://ionia.bu.edu/

Department of Electrical and Computer Engineering,
Division of Systems Engineering,
and Center for Information and Systems Engineering,
Boston University

Lecture 16: Outline

1. Lagrange multiplier theory.
2. Barrier methods.

Problems with equality constraints

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & h_i(\mathbf{x}) = 0, \qquad i = 1, \ldots, m \end{aligned}$$

where f, h_i are continuously differentiable (in a open set containing the minimum).
Letting $\mathbf{h} = (h_1, \ldots, h_m) : \mathbb{R}^n \to \mathbb{R}^m$ we have

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{aligned}$$

Necessary optimality conditions

Proposition

Let $\mathbf{x}^*$ be a local minimum. Assume that $\mathbf{x}^*$ is **regular**, i.e., $\nabla h_1(\mathbf{x}^*), \ldots, \nabla h_m(\mathbf{x}^*)$ are linearly independent. Then, there exists a Lagrange multiplier vector $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_m^*)$ s.t.

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(\mathbf{x}^*) = 0.$$

If $f, \mathbf{h}$ are twice continuously differentiable

$$\mathbf{y}' \left(\nabla^2 f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 h_i(\mathbf{x}^*) \right) \mathbf{y} \geq 0, \qquad \forall \mathbf{y} \in V(\mathbf{x}^*),$$

where $V(\mathbf{x}^*) = \{\mathbf{y} \mid \nabla h_i(\mathbf{x}^*)'\mathbf{y} = 0,\ i = 1, \ldots, m\}$ is the subspace of first order feasible directions.

Sufficient optimality conditions

Define the **Lagrangian Function**: $L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x})$.

Proposition

Let $\mathbf{x}^*, \boldsymbol{\lambda}^*$ satisfy

$$\nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0, \qquad \nabla_{\boldsymbol{\lambda}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0$$

$$\mathbf{y}' \nabla^2_{\mathbf{xx}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*) \mathbf{y} > 0, \qquad \forall \mathbf{y} \neq \mathbf{0} \text{ s.t. } \nabla \mathbf{h}(\mathbf{x}^*)' \mathbf{y} = \mathbf{0}.$$

Then $\mathbf{x}^*$ is a **strict** local min of f over $\mathbf{h}(\mathbf{x}) = \mathbf{0}$, i.e., $\exists \gamma, \epsilon > 0$ s.t.

$$f(\mathbf{x}) \geq f(\mathbf{x}^*) + \frac{\gamma}{2} ||\mathbf{x} - \mathbf{x}^*||^2, \qquad \forall \mathbf{x} \text{ s.t. } \mathbf{h}(\mathbf{x}) = \mathbf{0} \text{ with } ||\mathbf{x} - \mathbf{x}^*|| < \epsilon.$$

Problems with equality and inequality constraints

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), \ldots, h_m(\mathbf{x})) = \mathbf{0}, \\ & \mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_r(\mathbf{x})) \leq \mathbf{0}, \end{aligned}$$

where f, h_i, g_j are continuously differentiable (in a open set containing the minimum).

Define the **Lagrangian Function**:
$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i h_i(\mathbf{x}) + \sum_{j=1}^{r} \mu_j g_j(\mathbf{x})$.

Let $A(\mathbf{x}) = \{j \mid g_j(\mathbf{x}) = 0\}$.

Karush-Kuhn-Tucker necessary optimality conditions

Proposition

Let $\mathbf{x}^*$ be a regular local minimum, i.e., $\nabla h_1(\mathbf{x}^*), \ldots, \nabla h_m(\mathbf{x}^*), \nabla g_j(\mathbf{x}^*)$, $j \in A(\mathbf{x}^*)$ are linearly independent. Then, there exists Lagrange multiplier vectors $\boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_m^*)$ and $\boldsymbol{\mu}^* = (\mu_1^*, \ldots, \mu_r^*)$ s.t.

$$\nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0, \qquad \boldsymbol{\mu}^* \geq \mathbf{0},$$
$$\mu_j^* = 0, \qquad \forall j \notin A(\mathbf{x}^*)$$

If $f, \mathbf{h}, \mathbf{g}$ are twice continuously differentiable

$$\mathbf{y}' \nabla^2_{\mathbf{xx}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{y} \geq 0, \qquad \forall \mathbf{y} \in V(\mathbf{x}^*),$$

where $V(\mathbf{x}^*) = \{\mathbf{y} \mid \nabla h_i(\mathbf{x}^*)'\mathbf{y} = 0,\ \forall i,\ \nabla g_j(\mathbf{x}^*)'\mathbf{y} = 0,\ j \in A(\mathbf{x}^*)\}$.

Sufficient conditions

Proposition

Let $\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*$ satisfy

$$\nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) = 0, \qquad \mathbf{h}(\mathbf{x}^*) = \mathbf{0}, \qquad \mathbf{g}(\mathbf{x}^*) \leq \mathbf{0}$$
$$\boldsymbol{\mu}^* \geq \mathbf{0}, \qquad \mu_j^* = 0,\ \forall j \notin A(\mathbf{x}^*),$$
$$\mathbf{y}' \nabla^2_{\mathbf{xx}} L(\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \mathbf{y} > 0, \qquad \forall \mathbf{y} \neq \mathbf{0},\ \mathbf{y} \in V(\mathbf{x}^*),$$
$$\mu_j^* > 0,\ \forall j \in A(\mathbf{x}^*).$$

Then $\mathbf{x}^*$ is a strict local minimum of the constrained problem.

Barrier methods

Consider

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_r(\mathbf{x})) \leq \mathbf{0}, \\ & \mathbf{x} \in \mathscr{X}, \end{aligned}$$

and define the interior of the feasible set
$\mathscr{S} = \{\mathbf{x} \in \mathscr{X} \mid g_j(\mathbf{x}) < 0, \ \forall j\}$.

Barrier functions: for $\mathbf{x} \in \mathscr{S}$

$$B(\mathbf{x}) = -\sum_{j=1}^{r} \ln\left(-g_j(\mathbf{x})\right),$$

$$B(\mathbf{x}) = -\sum_{j=1}^{r} \frac{1}{g_j(\mathbf{x})}.$$

Barrier methods (cont.)

Barrier method:

$$\mathbf{x}^k = \arg\min_{\mathbf{x} \in \mathscr{S}} \{f(\mathbf{x}) + \epsilon^k B(\mathbf{x})\},$$

where $\{\epsilon^k\}$ is a sequence of scalars converging to zero ($\epsilon^k \to 0$) and satisfying

$$0 < \epsilon^{k+1} < \epsilon^k, \qquad k = 0, 1, \ldots$$

Theorem

Every limit point of $\{\mathbf{x}^k\}$ is a global minimum of the constrained problem.