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- 1 Introduction to Interior Point methods.
- 2 Affine scaling algorithm.
- 3 Potential reduction algorithm.

Interior Point Methods

- 1 Affine scaling.
- 2 Potential reduction.
- 3 Path following.

Affine Scaling

Consider an LP in standard form and its dual.

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad \begin{array}{ll} \max & \mathbf{p}'\mathbf{b} \\ \text{s.t.} & \mathbf{p}'\mathbf{A} \leq \mathbf{c}' \end{array}$$

Let $\beta \in (0, 1)$. Given $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} > \mathbf{0}$, $\mathbf{Ay} = \mathbf{b}$, create ellipsoid centered at \mathbf{y} in the interior.

$$S = \left\{ \mathbf{x} \mid \sum_{i=1}^n \frac{(y_i - x_i)^2}{y_i^2} \leq \beta^2 \right\}$$

$$S_0 = S \cap \{ \mathbf{x} \mid \mathbf{Ax} = \mathbf{b} \}$$

S_0 is ellipsoid centered at \mathbf{y} in the interior of feasible set P .

Affine Scaling (cont.)

Minimize over S_0 . Let $\mathbf{Y} = \text{diag}(y_1, \dots, y_n)$.

$$\begin{array}{l} \min \quad \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b} \\ \|\mathbf{Y}^{-1}(\mathbf{x} - \mathbf{y})\| \leq \beta \end{array} \quad \left\{ \begin{array}{l} \xrightarrow{\mathbf{d} = \mathbf{x} - \mathbf{y}} \\ \text{s.t.} \end{array} \right. \quad \begin{array}{l} \min \quad \mathbf{c}'\mathbf{d} \\ \mathbf{A}\mathbf{d} = \mathbf{0} \\ \|\mathbf{Y}^{-1}\mathbf{d}\| \leq \beta \end{array}$$

Assuming \mathbf{A} has lin. ind. rows and \mathbf{c} can not be written as their lin. combination we have

$$\begin{aligned} \mathbf{d}^* &= -\beta \frac{\mathbf{Y}^2(\mathbf{c} - \mathbf{A}'\mathbf{p})}{\|\mathbf{Y}(\mathbf{c} - \mathbf{A}'\mathbf{p})\|} \\ \mathbf{p} &= (\mathbf{A}\mathbf{Y}^2\mathbf{A}')^{-1}\mathbf{A}\mathbf{Y}^2\mathbf{c} \\ \mathbf{x} &= \mathbf{y} + \mathbf{d}^* \\ \mathbf{c}'\mathbf{x} &< \mathbf{c}'\mathbf{y} \end{aligned}$$

(Note that $\mathbf{d}^* \geq 0 \Rightarrow \text{cost} = -\infty$).

Affine Scaling (cont.)

\mathbf{p} has the interpretation of a *dual estimate*.

Let $\mathbf{r} = \mathbf{c} - \mathbf{A}'\mathbf{p}$. If $\mathbf{r} \geq \mathbf{0}$ then \mathbf{p} is dual feasible and

$$\mathbf{r}'\mathbf{y} = (\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{y} = \mathbf{c}'\mathbf{y} - \mathbf{p}'\mathbf{b}$$

is the duality gap.

When $\mathbf{r}'\mathbf{y} < \epsilon$ then \mathbf{y}, \mathbf{p} are near-optimal.

The affine scaling algorithm

- 1 (Initialization) Feasible $\mathbf{x}^0 > \mathbf{0}$, $k = 0$.
- 2 (Compute dual estimates and reduced costs) Given $\mathbf{x}^k > \mathbf{0}$

$$\begin{aligned} \mathbf{X}_k &= \text{diag}(x_1^k, \dots, x_n^k) \\ \mathbf{p}^k &= (\mathbf{A}\mathbf{X}_k^2\mathbf{A}')^{-1}\mathbf{A}\mathbf{X}_k^2\mathbf{c} \\ \mathbf{r}^k &= \mathbf{c} - \mathbf{A}'\mathbf{p}^k \end{aligned}$$

- 3 (Optimality check) If $\mathbf{r}^k \geq \mathbf{0}$ and $\mathbf{e}'\mathbf{X}_k\mathbf{r}^k < \epsilon$ then STOP; near optimal.
- 4 (Unboundedness) If $-\mathbf{X}_k^2\mathbf{r}^k \geq \mathbf{0}$ then optimal cost $= -\infty$.
- 5 (Update)

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \beta \frac{\mathbf{X}_k^2\mathbf{r}^k}{\|\mathbf{X}_k\mathbf{r}^k\|}$$

Potential Reduction

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad \begin{array}{ll} \max & \mathbf{p}'\mathbf{b} \\ \text{s.t.} & \mathbf{p}'\mathbf{A} + \mathbf{s}' = \mathbf{c}' \\ & \mathbf{s} \geq \mathbf{0} \end{array}$$

Potential Function $G(\mathbf{x}, \mathbf{s}) = q \log \mathbf{s}'\mathbf{x} - \sum_{j=1}^m \log x_j - \sum_{j=1}^n \log s_j$

Theorem

Start from $\mathbf{x}^0 > \mathbf{0}$, $(\mathbf{p}^0, \mathbf{s}^0)$ with $\mathbf{s}^0 > \mathbf{0}$ primal-dual feasible. If $G(\mathbf{x}, \mathbf{s})$ is reduced by at least δ at each iteration then after

$$K = \left\lceil \frac{G(\mathbf{x}^0, \mathbf{s}^0) + (q - n) \log \frac{1}{2} - n \log n}{\delta} \right\rceil$$

iterations we get $(\mathbf{s}^K)' \mathbf{x}^K \leq \epsilon$

The Potential Reduction Algorithm

- (Initialization) Start with feasible $\mathbf{x}^0 > \mathbf{0}$, $(\mathbf{p}^0, \mathbf{s}^0)$ with $\mathbf{s}^0 > \mathbf{0}$, set $k = 0$.
- (Optimality test) If $(\mathbf{s}^k)' \mathbf{x}^k < \epsilon$ STOP.
- (Compute update direction)

$$\begin{aligned} \mathbf{X}_k &= \text{diag}(x_1^k, \dots, x_n^k) \\ \bar{\mathbf{A}}^k &= (\mathbf{A}\mathbf{X}_k)'(\mathbf{A}\mathbf{X}_k^2\mathbf{A}')^{-1}\mathbf{A}\mathbf{X}_k \\ \mathbf{u}^k &= (\mathbf{I} - \bar{\mathbf{A}}^k) \left(\frac{q}{(\mathbf{s}^k)' \mathbf{x}^k} \mathbf{X}_k \mathbf{s}^k - \mathbf{e} \right) \\ \mathbf{d}^k &= -\beta \frac{\mathbf{X}_k \mathbf{u}^k}{\|\mathbf{u}^k\|} \end{aligned}$$

The Potential Reduction Algorithm (cont.)

- (Primal step) If $\|\mathbf{u}^k\| \geq \gamma$ then

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}^k, \quad \mathbf{s}^{k+1} = \mathbf{s}^k, \quad \mathbf{p}^{k+1} = \mathbf{p}^k$$

- (Dual step) If $\|\mathbf{u}^k\| < \gamma$ then

$$\mathbf{x}^{k+1} = \mathbf{x}^k, \quad \mathbf{s}^{k+1} = \frac{(\mathbf{s}^k)' \mathbf{x}^k}{q} (\mathbf{X}_k)^{-1} (\mathbf{u}^k + \mathbf{e})$$

$$\mathbf{p}^{k+1} = \mathbf{p}^k + (\mathbf{A}\mathbf{X}_k^2\mathbf{A}')^{-1} \mathbf{A}\mathbf{X}_k \left(\mathbf{X}_k \mathbf{s}^k - \frac{(\mathbf{s}^k)' \mathbf{x}^k}{q} \mathbf{e} \right)$$

- $k := k + 1$