SE524/EC524 Optimization Theory and Methods

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Lecture 17: Outline

1. Introduction to Interior Point methods.
2. Affine scaling algorithm.
3. Potential reduction algorithm.

Interior Point Methods

1. Affine scaling.
2. Potential reduction.
3. Path following.

Affine Scaling

Consider an LP in standard form and its dual.

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \begin{array}{ll} \max & \mathbf{p}'\mathbf{b} \\ \text{s.t.} & \mathbf{p}'\mathbf{A} \leq \mathbf{c}' \end{array}$$

Let $\beta \in (0,1)$. Given $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{y} > \mathbf{0}$, $\mathbf{A}\mathbf{y} = \mathbf{b}$, create ellipsoid centered at $\mathbf{y}$ in the interior.

$$S = \left\{ \mathbf{x} \,\middle|\, \sum_{i=1}^{n} \frac{(y_i - x_i)^2}{y_i^2} \leq \beta^2 \right\}$$

$$S_0 = S \cap \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}\}$$

S_0 is ellipsoid centered at $\mathbf{y}$ in the interior of feasible set P.

Affine Scaling (cont.)

Minimize over S_0. Let $\mathbf{Y} = \text{diag}(y_1, \ldots, y_n)$.

$$\left.\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & ||\mathbf{Y}^{-1}(\mathbf{x} - \mathbf{y})|| \leq \beta \end{array}\right\} \overset{\mathbf{d}=\mathbf{x}-\mathbf{y}}{\Longrightarrow} \begin{array}{ll} \min & \mathbf{c}'\mathbf{d} \\ \text{s.t.} & \mathbf{A}\mathbf{d} = \mathbf{0} \\ & ||\mathbf{Y}^{-1}\mathbf{d}|| \leq \beta \end{array}$$

Assuming $\mathbf{A}$ has lin. ind. rows and $\mathbf{c}$ can not be written as their lin. combination we have

$$\mathbf{d}^* = -\beta \frac{\mathbf{Y}^2(\mathbf{c} - \mathbf{A}'\mathbf{p})}{||\mathbf{Y}(\mathbf{c} - \mathbf{A}'\mathbf{p})||}$$

$$\mathbf{p} = (\mathbf{A}\mathbf{Y}^2\mathbf{A}')^{-1}\mathbf{A}\mathbf{Y}^2\mathbf{c}$$

$$\mathbf{x} = \mathbf{y} + \mathbf{d}^*$$

$$\mathbf{c}'\mathbf{x} < \mathbf{c}'\mathbf{y}$$

(Note that $\mathbf{d}^* \geq 0 \Rightarrow$ cost $= -\infty$).

Affine Scaling (cont.)

$\mathbf{p}$ has the interpretation of a *dual estimate*.

Let $\mathbf{r} = \mathbf{c} - \mathbf{A}'\mathbf{p}$. If $\mathbf{r} \geq \mathbf{0}$ then $\mathbf{p}$ is dual feasible and

$$\mathbf{r}'\mathbf{y} = (\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{y} = \mathbf{c}'\mathbf{y} - \mathbf{p}'\mathbf{b}$$

is the duality gap.

When $\mathbf{r}'\mathbf{y} < \epsilon$ then $\mathbf{y}, \mathbf{p}$ are near-optimal.

The affine scaling algorithm

1. (Initialization) Feasible $\mathbf{x}^0 > 0$, $k = 0$.
2. (Compute dual estimates and reduced costs) Given $\mathbf{x}^k > 0$

$$\mathbf{X}_k = \text{diag}(x_1^k, \dots, x_n^k)$$

$$\mathbf{p}^k = (\mathbf{A}\mathbf{X}_k^2\mathbf{A}')^{-1}\mathbf{A}\mathbf{X}_k^2\mathbf{c}$$

$$\mathbf{r}^k = \mathbf{c} - \mathbf{A}'\mathbf{p}^k$$

3. (Optimality check) If $\mathbf{r}^k \geq 0$ and $\mathbf{e}'\mathbf{X}_k\mathbf{r}^k < \epsilon$ then STOP; near optimal.
4. (Unboundedness) If $-\mathbf{X}_k^2\mathbf{r}^k \geq 0$ then optimal cost$= -\infty$.
5. (Update)

$$\mathbf{x}^{k+1} = \mathbf{x}^k - \beta\frac{\mathbf{X}_k^2\mathbf{r}^k}{||\mathbf{X}_k\mathbf{r}^k||}$$

Potential Reduction

$$\begin{array}{lc} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \begin{array}{lc} \max & \mathbf{p}'\mathbf{b} \\ \text{s.t.} & \mathbf{p}'\mathbf{A} + \mathbf{s}' = \mathbf{c}' \\ & \mathbf{s} \geq \mathbf{0} \end{array}$$

Potential Function $G(\mathbf{x}, \mathbf{s}) = q \log \mathbf{s}'\mathbf{x} - \sum_{j=1}^n \log x_j - \sum_{j=1}^n \log s_j$

Theorem

Start from $\mathbf{x}^0 > \mathbf{0}$, $(\mathbf{p}^0, \mathbf{s}^0)$ with $\mathbf{s}^0 > 0$ primal-dual feasible. If $G(\mathbf{x}, \mathbf{s})$ is reduced by at least δ at each iteration then after

$$K = \left\lceil \frac{G(\mathbf{x}^0, \mathbf{s}^0) + (q - n)\log\frac{1}{\epsilon} - n\log n}{\delta} \right\rceil$$

iterations we get $(\mathbf{s}^K)'\mathbf{x}^K \leq \epsilon$

The Potential Reduction Algorithm

- (Initialization) Start with feasible $\mathbf{x}^0 > \mathbf{0}$, $(\mathbf{p}^0, \mathbf{s}^0)$ with $\mathbf{s}^0 > 0$, set $k = 0$.
- (Optimality test) If $(\mathbf{s}^k)'\mathbf{x}^k < \epsilon$ STOP.
- (Compute update direction)

$$\mathbf{X}_k = \text{diag}(x_1^k, \ldots, x_n^k)$$

$$\bar{\mathbf{A}}^k = (\mathbf{A}\mathbf{X}_k)'(\mathbf{A}\mathbf{X}_k^2\mathbf{A}')^{-1}\mathbf{A}\mathbf{X}_k$$

$$\mathbf{u}^k = (\mathbf{I} - \bar{\mathbf{A}}^k)\left(\frac{q}{(\mathbf{s}^k)'\mathbf{x}^k}\mathbf{X}_k\mathbf{s}^k - \mathbf{e}\right)$$

$$\mathbf{d}^k = -\beta\frac{\mathbf{X}_k\mathbf{u}^k}{||\mathbf{u}^k||}$$

The Potential Reduction Algorithm (cont.)

- (Primal step) If $||\mathbf{u}^k|| \geq \gamma$ then

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}^k, \qquad \mathbf{s}^{k+1} = \mathbf{s}^k, \qquad \mathbf{p}^{k+1} = \mathbf{p}^k$$

- (Dual step) If $||\mathbf{u}^k|| < \gamma$ then

$$\mathbf{x}^{k+1} = \mathbf{x}^k, \qquad \mathbf{s}^{k+1} = \frac{(\mathbf{s}^k)'\mathbf{x}^k}{q}(\mathbf{X}_k)^{-1}(\mathbf{u}^k + \mathbf{e})$$

$$\mathbf{p}^{k+1} = \mathbf{p}^k + (\mathbf{A}\mathbf{X}_k^2\mathbf{A}')^{-1}\mathbf{A}\mathbf{X}_k\left(\mathbf{X}_k\mathbf{s}^k - \frac{(\mathbf{s}^k)'\mathbf{x}^k}{q}\mathbf{e}\right)$$

- $k := k + 1$