SE524/EC524 Optimization Theory and Methods

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Lecture 18: Outline

1. Path following algorithm.

Primal path following

Consider an LP in standard form and its dual.

$$\begin{array}{llcll} \min & \mathbf{c}'\mathbf{x} & \quad & \max & \mathbf{p}'\mathbf{b} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & & \text{s.t.} & \mathbf{p}'\mathbf{A} + \mathbf{s}' = \mathbf{c}' \\ & \mathbf{x} \geq \mathbf{0} & & & \mathbf{s} \geq \mathbf{0} \end{array}$$

Barrier function: $(\mu > 0)$ $\quad B_\mu(\mathbf{x}) = \mathbf{c}'\mathbf{x} - \mu \sum_{j=1}^{n} \log x_j$

Primal and dual barrier problems:

$$\begin{array}{llcll} \min & B_\mu(\mathbf{x}) & \quad & \max & \mathbf{p}'\mathbf{b} + \mu \sum_{j=1}^{n} \log s_j \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & & \text{s.t.} & \mathbf{p}'\mathbf{A} + \mathbf{s}' = \mathbf{c}' \end{array}$$

Let $\mathbf{x}(\mu)$ the optimal sol. of primal (central path)

$$\lim_{\mu \to 0} \mathbf{x}(\mu) = \mathbf{x}^*$$

The primal path following algorithm

1. (Initialization) Start with $\mathbf{x}^0 > \mathbf{0}$, $(\mathbf{p}^0, \mathbf{s}^0)$ with $\mathbf{s}^0 > 0$, set $k = 0$.
2. (Optimality test) If $(\mathbf{s}^k)'\mathbf{x}^k < \epsilon$ STOP.
3. $$\mathbf{X}_k = \text{diag}(x_1^k, \ldots, x_n^k)$$ $$\mu^{k+1} = \alpha \mu^k$$
4. (Compute directions) Solve linear system for $\mathbf{p}, \mathbf{d}$ $$\begin{array}{rcl} \mu^{k+1}\mathbf{X}_k^{-2}\mathbf{d} - \mathbf{A}'\mathbf{p} & = & \mu^{k+1}\mathbf{X}_k^{-1}\mathbf{e} - \mathbf{c} \\ \mathbf{A}\mathbf{d} & = & \mathbf{0} \end{array}$$
5. (Update) $$\mathbf{x}^{k+1} = \mathbf{x}^k + \mathbf{d}$$ $$\mathbf{p}^{k+1} = \mathbf{p}, \qquad \mathbf{s}^{k+1} = \mathbf{c} - \mathbf{A}'\mathbf{p}$$
6. $k := k + 1$

The primal-dual version

1. (Initialization) Start with $\mathbf{x}^0 > \mathbf{0}$, $(\mathbf{p}^0, \mathbf{s}^0)$ with $\mathbf{s}^0 > 0$, set $k = 0$.
2. (Optimality test) If $(\mathbf{s}^k)'\mathbf{x}^k < \epsilon$ STOP.
3. (Compute directions)

$$\mathbf{X}_k = \text{diag}(x_1^k, \ldots, x_n^k), \quad \mathbf{S}_k = \text{diag}(s_1^k, \ldots, s_n^k), \quad \mu^k = \frac{(\mathbf{s}^k)'\mathbf{x}^k}{n}$$

   Solve linear system for $\mathbf{d} = (\mathbf{d}_x^k, \mathbf{d}_p^k, \mathbf{d}_s^k)$.
4. (Set step sizes)

$$\beta_P^k = \min\left\{1, \alpha \min_{\{i|(d_x^k)_i<0\}} \left(-\frac{x_i^k}{(d_x^k)_i}\right)\right\}, \beta_D^k = \min\left\{1, \alpha \min_{\{i|(d_s^k)_i<0\}} \left(-\frac{s_i^k}{(d_s^k)_i}\right)\right\}$$

5. (Update)

$$\mathbf{x}^{k+1} = \mathbf{x}^k + \beta_P^k \mathbf{d}_x^k, \quad \mathbf{p}^{k+1} = \mathbf{p}^k + \beta_D^k \mathbf{d}_p^k, \quad \mathbf{s}^{k+1} = \mathbf{s}^k + \beta_D^k \mathbf{d}_s^k$$

6. $k := k + 1$

Practical Guidelines

- Simplex does badly on large massively degenerate problems. Interior point are the methods of choice here.
- For interior point methods it is critical how we solve the linear systems of equations that arise (use sophisticated method like Cholesky factorization, exploit sparsity of matrices).