SE524/EC524 Optimization Theory and Methods

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Lecture 19: Outline

1. Graphs and definitions.
2. Network flow problems.
3. Network simplex.

Undirected Graphs

- **Undirected** $\mathscr{G} = (\mathscr{N}, \mathscr{E})$
  - $\mathscr{N}$: nodes
  - $\mathscr{E} \ni (i,j)$: arcs/edges, unordered pair of nodes. (i,j) is incident to nodes i,j which are called endpoints of the arc.
  - No self-arcs
  - **degree of node** i: $\#$ of arcs incident to i
  - **degree of** $\mathscr{G}$: max of node degrees
  - **walk**: finite sequence of nodes $i_1, \ldots, i_t$ connected with arcs
  - **path**: walk with no repeated nodes.
  - **cycle**: a walk $i_1, \ldots, i_t$ so that $i_1, \ldots, i_{t-1}$ is a path, $i_1 = i_t$, and $t - 1 \geq 3$
  - **connected graph**: $\forall i,j \in \mathscr{N}$ $\exists$ path $i \to j$.

Directed Graphs

- **Directed** $\mathscr{G} = (\mathscr{N}, \mathscr{A})$
  - $\mathscr{N}$: nodes
  - $\mathscr{A} \ni (i,j)$: arcs/edges, ordered pair of nodes. (i,j) outgoing from i, incoming to j, incident to both i,j.
  - No self-arcs
  - **Incoming to** i: $I(i) = \{j \in \mathscr{N} \mid (j,i) \in \mathscr{A}\}$.
  - **Outgoing from** i: $O(i) = \{j \in \mathscr{N} \mid (i,j) \in \mathscr{A}\}$.
  - **walk**: finite sequence of nodes $i_1, \ldots, i_t$ with associated arcs $a_1, \ldots, a_{t-1}$ such that $\forall k = 1, \ldots, t-1$ either $a_k = (i_k, i_{k+1})$ **(forward arc)** or $a_k = (i_{k+1}, i_k)$ **(backward arc)**.
  - **path**: walk with no repeated nodes.
  - **cycle**: a walk $i_1, \ldots, i_t$ so that $i_1, \ldots, i_{t-1}$ is a path, $i_1 = i_t$. (Now, we allow a cycle that involves two nodes).
  - **directed walk or cycle**: if all arcs are forward arcs.
  - **connected directed graph**: If the corresponding undirected graph (ignore directions and remove duplicate arcs) is connected.

Trees

Trees: Undirected graph $\mathscr{G} = (\mathscr{N}, \mathscr{E})$ that is connected and has no cycles. A node is called a **leaf** if its degree is 1.

Theorem

1. Every tree with more than one node has at least one leaf.
2. An undirected graph is a tree **iff** is connected and has $|\mathscr{N}| - 1$ arcs.
3. $\exists$ a unique path from each node i to each node j in the tree.
4. If we add a new arc to a tree the resulting graph contains exactly one cycle.

Spanning Trees

Consider a connected undirected graph $\mathscr{G} = (\mathscr{N}, \mathscr{E})$ and let $\mathscr{E}_1 \subseteq \mathscr{E}$ such that $\mathscr{T} = (\mathscr{N}, \mathscr{E}_1)$ is a tree. $\mathscr{T}$ is called a **spanning tree**.

Theorem

Let $\mathscr{G} = (\mathscr{N}, \mathscr{E})$ a connected undirected graph and let $\mathscr{E}_0 \subseteq \mathscr{E}$ such that arcs in $\mathscr{E}_0$ do not form any cycles. Then $\mathscr{E}_0$ can be augmented to $\mathscr{E}_1 \supseteq \mathscr{E}_0$ such that $(\mathscr{N}, \mathscr{E}_1)$ is a spanning tree.

Network flow problems

- Directed graph $\mathscr{G} = (\mathscr{N}, \mathscr{A})$
- b_i: external supply to node i
- u_{ij}: capacity of arc (i, j)
- c_{ij}: cost per unit of flow on arc (i, j)

Formulation: **flow vector** $\mathbf{f} = (f_{ij};\ (i,j) \in \mathscr{A})$.

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in\mathscr{A}} c_{ij} f_{ij} \\
\text{s.t.} \quad & b_i + \sum_{j \in I(i)} f_{ji} = \sum_{j \in O(i)} f_{ij}, \qquad \forall i \in \mathscr{N}, \\
& 0 \leq f_{ij} \leq u_{ij}, \qquad \forall (i,j) \in \mathscr{A}.
\end{aligned}
$$

Some special cases: **transportation problem**, **assignment problem**.

Incidence matrix formulation

- $\mathscr{N} = \{1, \ldots, n\}$
- $|\mathscr{A}| = m$
- **incidence matrix** $n \times m$ matrix

$$
\mathbf{A} = (a_{ik}) \qquad a_{ik} = \begin{cases} 1 & \text{if } i \text{ is the start of the } k\text{th arc} \\ -1 & \text{if } i \text{ is the end of the } k\text{th arc} \\ 0 & \text{otherwise} \end{cases}
$$

- **Flow conservation**: $\mathbf{A}\mathbf{f} = \mathbf{b}$.
- **circulation** $\mathbf{f}$ that satisfies $\mathbf{A}\mathbf{f} = \mathbf{0}$.
- Rows of $\mathbf{A}$ add up to the zero vector $\Rightarrow$ linearly dependent.
- Let $\tilde{\mathbf{A}}$ denote the matrix of the first $n-1$ rows of $\mathbf{A}$ and $\tilde{\mathbf{b}}$ the vector with the first $n-1$ elements of $\mathbf{b}$.

Network simplex

Assume: $\sum_{i \in \mathcal{N}} b_i = 0$ and $\mathcal{G}$ is connected.
Consider uncapacitated problem:

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{f} \\ \text{s.t.} \quad & \tilde{\mathbf{A}}\mathbf{f} = \tilde{\mathbf{b}} \\ & \mathbf{f} \geq \mathbf{0} \end{aligned}$$

- **BFS**: tree solutions.
- Change of basis.
- Computation of reduced costs.
- Initialization.
- Integrality.
- Generalization to capacitated problems.