Graphs and definitions.

Network flow problems.

Network simplex.
Undirected Graphs

- **Undirected** $\mathcal{G} = (\mathcal{N}, \mathcal{E})$
  - $\mathcal{N}$: nodes
  - $\mathcal{E} \ni (i, j)$: arcs/edges, unordered pair of nodes. $(i, j)$ is incident to nodes $i, j$ which are called endpoints of the arc.
  - No self-arcs
  - **degree of node** $i$: # of arcs incident to $i$
  - **degree of** $\mathcal{G}$: max of node degrees
  - **walk**: finite sequence of nodes $i_1, \ldots, i_t$ connected with arcs
  - **path**: walk with no repeated nodes.
  - **cycle**: a walk $i_1, \ldots, i_t$ so that $i_1, \ldots, i_{t-1}$ is a path, $i_1 = i_t$, and $t - 1 \geq 3$
  - **connected graph**: $\forall i, j \in \mathcal{N} \exists$ path $i \rightarrow j$.

Directed Graphs

- **Directed** $\mathcal{G} = (\mathcal{N}, \mathcal{A})$
  - $\mathcal{N}$: nodes
  - $\mathcal{A} \ni (i, j)$: arcs/edges, ordered pair of nodes. $(i, j)$ outgoing from $i$, incoming to $j$, incident to both $i, j$.
  - No self-arcs
  - **Incoming to** $i$: $I(i) = \{ j \in \mathcal{N} | (j, i) \in \mathcal{A} \}$.
  - **Outgoing from** $i$: $O(i) = \{ j \in \mathcal{N} | (i, j) \in \mathcal{A} \}$.
  - **walk**: finite sequence of nodes $i_1, \ldots, i_t$ with associated arcs $a_1, \ldots, a_{t-1}$ such that $\forall k = 1, \ldots, t - 1$ either $a_k = (i_k, i_{k+1})$ (forward arc) or $a_k = (i_{k+1}, i_k)$ (backward arc).
  - **path**: walk with no repeated nodes.
  - **cycle**: a walk $i_1, \ldots, i_t$ so that $i_1, \ldots, i_{t-1}$ is a path, $i_1 = i_t$.
  - (Now, we allow a cycle that involves two nodes).
  - **directed walk or cycle**: if all arcs are forward arcs.
  - **connected directed graph**: If the corresponding undirected graph (ignore directions and remove duplicate arcs) is connected.
Trees

**Trees:** Undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ that is connected and has no cycles. A node is called a **leaf** if its degree is 1.

**Theorem**

1. Every tree with more than one node has at least one leaf.
2. An undirected graph is a tree iff it is connected and has $|\mathcal{N}| - 1$ arcs.
3. $\exists$ a unique path from each node $i$ to each node $j$ in the tree.
4. If we add a new arc to a tree the resulting graph contains exactly one cycle.

Spanning Trees

Consider a connected undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ and let $\mathcal{E}_1 \subseteq \mathcal{E}$ such that $\mathcal{T} = (\mathcal{N}, \mathcal{E}_1)$ is a tree. $\mathcal{T}$ is called a **spanning tree**.

**Theorem**

Let $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ a connected undirected graph and let $\mathcal{E}_0 \subseteq \mathcal{E}$ such that arcs in $\mathcal{E}_0$ do not form any cycles. Then $\mathcal{E}_0$ can be augmented to $\mathcal{E}_1 \supseteq \mathcal{E}_0$ such that $(\mathcal{N}, \mathcal{E}_1)$ is a spanning tree.
Network flow problems

- Directed graph $G = (\mathcal{N}, \mathcal{A})$
- $b_i$: external supply to node $i$
- $u_{ij}$: capacity of arc $(i, j)$
- $c_{ij}$: cost per unit of flow on arc $(i, j)$

Formulation: flow vector $f = (f_{ij}; (i, j) \in \mathcal{A})$.

$$\begin{align*}
\min & \sum_{(i, j) \in \mathcal{A}} c_{ij} f_{ij} \\
\text{s.t.} & \quad b_i + \sum_{j \in I(i)} f_{ji} = \sum_{j \in O(i)} f_{ij}, \quad \forall i \in \mathcal{N}, \\
& \quad 0 \leq f_{ij} \leq u_{ij}, \quad \forall (i, j) \in \mathcal{A}.
\end{align*}$$

Some special cases: transportation problem, assignment problem.

Incidence matrix formulation

- $\mathcal{N} = \{1, \ldots, n\}$
- $|\mathcal{A}| = m$
- incidence matrix $n \times m$ matrix

$$A = (a_{ik}) \quad a_{ik} = \begin{cases} 1 & \text{if } i \text{ is the start of the } k\text{th arc} \\ -1 & \text{if } i \text{ is the end of the } k\text{th arc} \\ 0 & \text{otherwise} \end{cases}$$

- Flow conservation: $Af = b$.
- circulation $f$ that satisfies $Af = 0$.
- Rows of $A$ add up to the zero vector $\Rightarrow$ linearly dependent.
- Let $\tilde{A}$ denote the matrix of the first $n - 1$ rows of $A$ and $\tilde{b}$ the vector with the first $n - 1$ elements of $b$. 
Network simplex

Assume: $\sum_{i \in N} b_i = 0$ and $\mathcal{G}$ is connected.
Consider uncapacitated problem:

$$\begin{align*}
\min \quad & c'f \\
\text{s.t.} \quad & \tilde{A}f = \tilde{b} \\
\quad & f \geq 0
\end{align*}$$

- **BFS**: tree solutions.
- Change of basis.
- Computation of reduced costs.
- Initialization.
- Integrality.
- Generalization to capacitated problems.