SE524/EC524 Optimization Theory and Methods

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Lecture 2: Outline

1. Applications of LP and formulations.
2. Nonlinear objective functions.
3. Linear Algebra Review.
4. Geometric concepts: polyhedra and convex sets.

Examples and formulations

How to formulate ? There is no systematic method ...

1. Think to define decision variables.
2. Translate objective and problem data to constraints and objective function.
3. **Be parsimonious !**

Some formulations:

- The Diet Problem.
- Production Planning in a manufacturing plant.
- Routing in a communication network.
- Maximum lifetime routing in wireless sensor networks.
- Flux balance analysis in metabolic networks.

Piecewise linear convex functions

$$\begin{array}{ll} \min & \max_{i=1,\ldots,m}(\mathbf{c}_i'\mathbf{x} + d_i) \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \max_{j=1,\ldots,k}(\mathbf{f}_j'\mathbf{x} + g_j) \leq h \end{array}$$

equivalent to LP

$$\begin{array}{ll} \min & z \\ \text{s.t.} & z \geq \mathbf{c}_i'\mathbf{x} + d_i, \quad i = 1, \ldots, m \\ & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{f}_j'\mathbf{x} + g_j \leq h, \quad j = 1, \ldots, k \end{array}$$

Absolute values

$$\begin{array}{ll} \min & \sum_{i=1}^{n} c_i|x_i| \qquad (c_i \geq 0) \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{array}$$

equivalent to LP

$$\begin{array}{ll} \min & \sum_{i=1}^{n} c_i z_i \\ \text{s.t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & x_i \leq z_i \\ & -x_i \leq z_i \end{array}$$

and also equivalent to LP

$$\begin{array}{ll} \min & \sum_{i=1}^{n} c_i(x_i^{+} + x_i^{-}) \\ \text{s.t.} & \mathbf{A}(\mathbf{x}^{+} - \mathbf{x}^{-}) \geq \mathbf{b} \\ & \mathbf{x}^{+}, \mathbf{x}^{-} \geq \mathbf{0} \end{array}$$

Proof as Exercise ...

Polyhedral Geometry: Some Definitions

Polyhedron is a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$.

Polyhedron in standard form is a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$.

Polytope is a bounded polyhedron (i.e., $\exists K$ s.t. $|x_i| \leq K$ $\forall \mathbf{x} = (x_1, \ldots, x_n) \in$ polyhedron).

Hyperplane is a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}'\mathbf{x} = b\}$.

Halfspace is a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}'\mathbf{x} \geq b\}$.

Polyhedral Geometry: Some Definitions (cont.)

A polyhedron is the intersection of a finite number of halfspaces ($\mathbf{Ax} \geq \mathbf{b} \Rightarrow \mathbf{a_i'x} \geq b_i$, $i = 1, \ldots, m$).

$S \subset \mathbb{R}^n$ is **convex** if for any $\mathbf{x}, \mathbf{y} \in S$ and any $\lambda \in [0, 1]$:
$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in S$.
Let $\lambda_i \geq 0$ $\sum_{i=1}^{k} \lambda_i = 1$. The vector $\sum_{i=1}^{k} \lambda_i \mathbf{x_i}$ is said to be **convex combination** of $\mathbf{x_i}$'s.

Convex hull of $\mathbf{x_i}$'s is the set of all their convex combinations.

Properties of Convex Sets

Theorem

1. The intersection of convex sets is convex.
2. Polyhedra are convex sets.
3. Convex combination of finite number of elements of a convex set belongs to the set.
4. Convex hull of a finite number of vectors is a convex set.