SE524/EC524 Optimization Theory and Methods

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Lecture 20: Outline

1. Maximum flow problem.
2. Network flow duality.
3. Shortest path problem.
4. Minimum spanning trees.

Maximum flow problem

- Directed $\mathscr{G} = (\mathscr{N}, \mathscr{A})$
- capacities u_{ij} for $(i,j) \in \mathscr{A}$.
- Source node s and sink node t.
- **Objective**: Ship the largest possible flow from s to t.

Formulation:

$$\begin{aligned} \max \quad & b_s \\ \text{s.t.} \quad & \mathbf{Af} = \mathbf{b} \\ & b_t = -b_s \\ & b_i = 0, \qquad \forall i \neq s, t \\ & 0 \leq f_{ij} \leq u_{ij}, \qquad \forall (i,j) \in \mathscr{A} \end{aligned}$$

Decision variables: $\mathbf{f}, \mathbf{b}$.

Network flow duality

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{f} \\ \text{s.t.} \quad & \mathbf{Af} = \mathbf{b} \\ & \mathbf{f} \geq \mathbf{0} \end{aligned} \qquad \begin{aligned} \max \quad & \mathbf{p}'\mathbf{b} \\ \text{s.t.} \quad & \mathbf{p}'\mathbf{A} \leq \mathbf{c}' \end{aligned}$$

and since $\mathbf{A}$ is an incidence matrix

$$\begin{aligned} \max \quad & \mathbf{p}'\mathbf{b} \\ \text{s.t.} \quad & p_i - p_j \leq c_{ij}, \qquad \forall (i,j) \in \mathscr{A} \end{aligned}$$

- Sensitivity.
- Complementary slackness.

Shortest path problem

- Directed $\mathscr{G} = (\mathscr{N}, \mathscr{A})$, n nodes, m arcs.
- c_{ij}: cost of arc (i, j).
- **Objective**: Find shortest (directed) path from node i to node j.

Network flow formulation: Consider all-to-one problem with destination node n. Assume $\exists$ directed path from each node to node n. Set

- $b_i = 1$, $i = 1, \ldots, n-1$
- $b_n = -(n-1)$ (demand at node n)
- uncapacitated problem

Shortest path problem (cont.)

Dual problem: $b_i = 1$, $i = 1, \ldots, n-1$; $p_n = 0$.

$$\begin{aligned} \max \quad & \sum_{i=1}^{n-1} p_i \\ \text{s.t.} \quad & p_i \leq c_{ij} + p_j, \qquad \forall (i,j) \in \mathscr{A} \end{aligned}$$

An optimal solution $\mathbf{p}^*$ satisfies **Bellman's** equations:

$$p_i^* = \min_{k \in O(i)} \{c_{ik} + p_k^*\}$$

Value iteration leads to Bellman-Ford algorithm.