Lecture 22: Outline

- Families of Integer Programming methods.
- Cutting plane methods.
- Branch and bound.
- Dynamic programming: Traveling salesman problem.
- IP duality.
- Approximation algorithms.
- Heuristics.
Integer Programming methods

- **Exact Methods** (EXP-time): cutting plane, branch and bound, DP.
- **Approximation Methods** (POLY-time): suboptimal solutions.
- **Heuristic Methods**: local search, simulated annealing.

### Cutting Plane methods

\[
\begin{align*}
\text{min} & \quad c'x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0 \\
& \quad x : \text{integer}
\end{align*}
\]

**Cutting plane algorithm:**

1. Solve LP relaxation of IP \( \Rightarrow x^* \).
2. If \( x^* \) integer STOP.
3. Otherwise “cut” \( x^* \): add violating constraint which is satisfied by all IP feasible solutions.

How do we cut? **Gomory cuts.**
Branch and Bound

- Solve LP relaxation ⇒ \( x^* \). Select fractional component \( x_j^* \).
- Create two subproblems \( F_1, F_2 \) by adding either of:

  \[ x_i \leq \lfloor x_i^* \rfloor \quad \text{or} \quad x_i \geq \lceil x_i^* \rceil \]

  Make both active.
- LP relaxation of each subproblem \( F_i \) yields lower bound \( b(F_i) \).
- Maintain upper bound \( U \) on IP optimal cost (\( U \) obtained by evaluating the cost of an IP feasible sol., e.g., when a subproblem has an integer optimal sol.)

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Branch and Bound algorithm

- Initialize \( U = \infty \).
- Select an active subproblem.
- Consider LP relaxation. If infeasible delete subproblem. Else solve.
- If optimal solution integer delete subproblem and update \( U \). Else compute \( b(F_i) \).
- If \( b(F_i) \geq U \), delete subproblem.
- If \( b(F_i) < U \) divide into two further subproblems.
Approximation algorithms

**Definition**

For a minimization problem: An algorithm is an $\epsilon$-approximation algorithm if it runs in POLY-time and returns a feasible solution $Z$ such that

$$Z \leq (1 + \epsilon)Z^*$$

For maximization:

$$Z \geq (1 - \epsilon)Z^*$$