SE524/EC524 Optimization Theory and Methods

Yannis Paschalidis
yannisp@bu.edu, http://ionia.bu.edu/

Department of Electrical and Computer Engineering,
Division of Systems Engineering,
and Center for Information and Systems Engineering,
Boston University

Lecture 3: Outline

1. Corner Points:
   1. Extreme point.
   2. Vertex.
   3. Basic Feasible Solution (BFS).
2. Theorem: Extreme point $\Leftrightarrow$ Vertex $\Leftrightarrow$ BFS.
3. BFS in standard form polyhedra.
4. Degeneracy is standard form polyhedra.

Extreme Points

Definition

Let P a polyhedron. A vector $\mathbf{x} \in P$ is an **extreme point** if $\nexists$ $\mathbf{y}, \mathbf{z} \in P$ and $\neq \mathbf{x}$ s.t. $\mathbf{x}$ can be written as a convex combination of $\mathbf{y}, \mathbf{z}$.

Vertex

Definition

A vector $\mathbf{x} \in P$ is an **vertex** if $\exists$ $\mathbf{c}$ s.t. $\mathbf{x}$ is the unique minimizer of $\mathbf{c}'\mathbf{y}$ over P.

Basic Feasible Solution (BFS)

We call **active** or **binding** the constraints that are satisfied with equality.

Definition

Let $\mathbf{x}^* \in \mathbb{R}^n$, and a polyhedron P.

1. $\mathbf{x}^*$ is a **basic solution** if
   1. All equality constraints are active.
   2. Let $I = \{i \mid \mathbf{a}_i'\mathbf{x}^* = \mathbf{b}_i\}$. The subspace spanned by $\{\mathbf{a}_i \mid i \in I\}$ is all of $\mathbb{R}^n$.
2. If $\mathbf{x}^*$ is feasible then is called a **basic feasible solution (bfs)**.

A basic solution $\mathbf{x}^*$ is called **degenerate** if $|I| > n$, and **nondegenerate** otherwise.

Equivalence of corner point definitions

Theorem

Extreme point $\Leftrightarrow$ Vertex $\Leftrightarrow$ BFS.

Proof:

1. Vertex $\Rightarrow$ Extreme point.
2. Extreme point $\Rightarrow$ BFS.
3. BFS $\Rightarrow$ Vertex.

□

- Two basic solutions are **adjacent** if there are $n-1$ linearly independent constraints active at both of them.
- If two bfs are adjacent, the line segment that connects them is called an **edge**.

BFS in standard form polyhedra

Let $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ nonempty.
WLOG assume $\mathbf{A}$ has m linearly independent rows.
Example

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 2x_1 + x_2 &= 4 \\ 2x_1 + 4x_2 &= 4 \end{aligned}$$

For a BFS we need n active "lin. ind." constraints: $\mathbf{A}\mathbf{x} = \mathbf{b}$ provides m. Hence, $n - m$ of $\mathbf{x} \geq \mathbf{0}$ should be active.

BFS in standard form polyhedra: Construction

Pick m linearly independent columns of $\mathbf{A}$ (wlog $1, 2, \ldots, m$).

$$m\{ [\overbrace{\mathbf{B}}^{m} \mid \overbrace{\mathbf{N}}^{n-m}] \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix} = \mathbf{b} \quad \Rightarrow \quad \begin{cases} \mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{x_N} = \mathbf{0} \end{cases}$$

- $\mathbf{x}^* = (\mathbf{x_B}, \mathbf{x_N})$ is a **basic solution**.
- If $\mathbf{x}^* \geq \mathbf{0}$ it is a **basic feasible solution**.
- The elements of $\mathbf{x_B}$ are called **basic variables**.
- The elements of $\mathbf{x_N}$ are called **nonbasic variables**.
- $\mathbf{B}$ is called a **basis matrix**, its columns form a **basis** (of $\mathbb{R}^m$).
- Two bases that share $m - 1$ columns are called **adjacent** and they correspond to adjacent bfs.
- A basis **uniquely** determines a bfs, **but** different bases may lead to the same bfs.

Degeneracy is standard form polyhedra

Definition

When the set of active constraints at a BFS has more than n elements then the BFS is **degenerate**.

In **standard form**: $\mathbf{Ax} = \mathbf{b}$ are m active constraints. If more that $n - m$ of $\mathbf{x} \geq \mathbf{0}$ active then **degenerate bfs**.

The following statements are equivalent and imply **degeneracy**:

- More than $n - m$ variables $= 0$.
- Some basic variables $= 0$.