# SE524/EC524 Optimization Theory and Methods

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## Lecture 4: Outline

1. Existence of extreme points.
2. Optimality of extreme points.
3. On the running time of algorithms and some complexity theory.

## Existence of Extreme Points

A polyhedron $P \subset \mathbb{R}^n$ **contains a line** if $\exists\, \mathbf{x} \in P$ and $\mathbf{d} \in \mathbb{R}^n$ ($\mathbf{d} \neq \mathbf{0}$) s.t. $\mathbf{x} + \lambda \mathbf{d} \in P$, $\forall \lambda \in \mathbb{R}$.

**Theorem**

Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a_i'x} \geq \mathbf{b_i},\ i = 1, \ldots, m\} \neq \varnothing$. Then

$P$ **has a BFS** $\Leftrightarrow$ $P$ **contains no line**

Special cases:

- Bounded Polyhedra **do not** contain a line $\Rightarrow$ **have a BFS**.
- Standard form polyhedra are in the positive orthant $\Rightarrow$ do not contain a line $\Rightarrow$ have a BFS.





## Optimality of extreme points

Consider the LP of minimizing $\mathbf{c'x}$ over a polyhedron $P \neq \varnothing$. WLOG assume $P$ has a bfs (otherwise can write LP in standard form and then $P$ will have a bfs).

**Theorem**

**(The central LP thm.)** Either

- Optimal cost $= -\infty$, or
- $\exists$ a bfs which is optimal.

**Proof**:Start from an arbitrary $\mathbf{x} \in P$. We can keep decreasing the cost by moving from point to point until we hit a bfs (otherwise the cost is $-\infty$). $\square$

## Implications

**Remarks**

- Previous thm. says that to find the optimum it suffices to consider only extreme points. This is what the **simplex method** does. Moving from bfs to bfs until it finds an optimum.
- In LP it is holds that **local minima** are also **global minima**, which is a far more general property. We will see that it holds in **convex programming** (optimization of convex function over a convex set). LP is a special case of convex programming.





## Notation for the Running time of algorithms

Let $f, g : \mathbb{R}_+ \to \mathbb{R}_+$

**Definition**

$f(n) = O(g(n))$ if $\exists n_0, c \geq 0$ such that $f(n) \leq cg(n)$ $\forall n \geq n_0$.

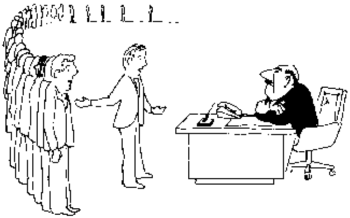
**Definition**

$f(n) = \Omega(g(n))$ if $\exists n_0, c \geq 0$ such that $f(n) \geq cg(n)$ $\forall n \geq n_0$.

**Definition**

$f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ **and** $f(n) = \Omega(g(n))$ hold.

## Complexity Theory



I can't find an efficient algorithm, but neither can all these famous people.

(from Garey & Johnson, 1979)





## Complexity Theory (cont.)

**Definition**

(**Reductions**) Let $\Pi_1, \Pi_2$ two recognition problems. We say $\Pi_1$ **transforms** or **reduces** to $\Pi_2$ in polynomial time if there exists a poly-time algorithm which given an instance $I_1$ of $\Pi_1$ outputs an instance $I_2$ of $\Pi_2$ with the property: $I_1$ is a YES **iff** $I_2$ is a YES.

**Definition**

(**NP-hard**) A problem is NP-hard if ZOIP can be transformed to it in poly-time.

**Definition**

(**Belongs to NP**) A problem belongs to NP if it can be transformed to ZOIP in poly-time.

**Definition**

(**NP-complete**) A problem is NP-complete if it belongs to NP and is also NP-hard.