SE524/EC524 Optimization Theory and Methods

Yannis Paschalidis
yannisp@bu.edu, http://ionia.bu.edu/

Department of Electrical and Computer Engineering,
Division of Systems Engineering,
and Center for Information and Systems Engineering,
Boston University

Lecture 5: Outline

1. Moving from BFS to BFS.
2. Bookkeeping: The Simplex Tableau.

Moving from BFS to BFS

Consider a **standard form** polyhedron $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ ($\mathbf{A}$ has linearly independent rows).

1. Find a bfs $\mathbf{x}$. Let the basis matrix

$$\mathbf{B} = \begin{bmatrix} | & & | \\ \mathbf{A}_{B(1)} & \cdots & \mathbf{A}_{B(m)} \\ | & & | \end{bmatrix} \quad \text{and} \quad \mathbf{x}_B = \begin{bmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b}$$

2. Let $j \notin \{B(1), \ldots, B(m)\}$. **Bring** x_j into the basis: $\mathbf{x} + \theta\mathbf{d}$ ($d_j = 1,\ d_k = 0,\ k \neq B(1), \ldots, B(m), j$) while retaining feasibility

$$\mathbf{A}_j = \sum_{i=1}^{m} x_{ij}\mathbf{A}_{B(i)} = \mathbf{B}\begin{bmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{bmatrix} \Rightarrow \begin{bmatrix} x_{1j} \\ \vdots \\ x_{mj} \end{bmatrix} = \mathbf{B}^{-1}\mathbf{A}_j$$

Moving from BFS to BFS (cont.)

How are basic variables affected ?

$$\mathbf{b} = \sum_{i=1}^{m} x_{B(i)}\mathbf{A}_{B(i)}$$
$$= \theta\mathbf{A}_j + \sum_{i=1}^{m} x_{B(i)}\mathbf{A}_{B(i)} - \theta\sum_{i=1}^{m} x_{ij}\mathbf{A}_{B(i)}$$

How do we move ? As $\theta \nearrow$

j	$x_j = 0$	$\rightarrow$	$x_j = \theta$
basic	$\mathbf{x}_B = \begin{bmatrix} \vdots \\ x_{B(i)} \\ \vdots \end{bmatrix}$	$\rightarrow$	$\mathbf{x}_B - \theta\mathbf{B}^{-1}\mathbf{A}_j = \begin{bmatrix} \vdots \\ x_{B(i)} - \theta x_{ij} \\ \vdots \end{bmatrix}$
nonbasic	$x_k = 0$	$\rightarrow$	$x_k = 0 \quad k \notin \{B(1), \ldots, B(m), j\}$

Moving from BFS to BFS (cont.)

How far can we move ? As long as $\mathbf{x} \geq \mathbf{0}$ holds, i.e.,

$$\theta^* = \min_{\{i | x_{ij} > 0\}} \frac{x_{B(i)}}{x_{ij}} \triangleq \frac{x_{B(l)}}{x_{lj}}$$

Let l the minimizer, then lth basic variable **exits the basis**.

Old basis: $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(l)}, \ldots, \mathbf{A}_{B(m)}$
New basis: $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(l-1)}, \mathbf{A}_j, \mathbf{A}_{B(l+1)}, \ldots, \mathbf{A}_{B(m)}$

Remarks

1. $x_{ij} \leq 0, \ \forall i$. Move along a half-line.
2. $\exists \ i$ s.t. $x_{B(i)} = 0, x_{ij} > 0$. Then $\mathbf{x}$ is degenerate, new bfs the same, have only changed basis.
3. Tie in definition of θ^*. Have moved to a degenerate bfs.

Algorithm: Moving from bfs to bfs

1. **Start** with basic columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ and associated bfs $\mathbf{x}$.
2. **Bring x_j into basis**:
   Compute $\mathbf{u} = (x_{1j}, \ldots, x_{mj})' = \mathbf{B}^{-1}\mathbf{A}_j$. If no component positive, then $\theta^* = \infty$ and cost $= \pm\infty$.
3. If some component positive, let

$$\theta^* = \min_{\{i | x_{ij} > 0\}} \frac{x_{B(i)}}{x_{ij}}$$

4. Let l the minimizer. Form a new basis by replacing $\mathbf{A}_{B(l)}$ with $\mathbf{A}_j$. If $\mathbf{y}$ the new bfs, then basic variables have values:

$$y_j = \theta^*$$
$$y_{B(i)} = x_{B(i)} - \theta^* x_{ij}, \quad i \neq l$$

Simplex tableau

$$[\mathbf{B}^{-1}\mathbf{b} \quad | \quad \mathbf{B}^{-1}\mathbf{A}_1 \ldots \mathbf{B}^{-1}\mathbf{A}_n] \quad = \quad \left[\begin{array}{c|ccc} x_{B(1)} & x_{11} & \ldots & x_{1n} \\ \vdots & \vdots & & \vdots \\ x_{B(m)} & x_{m1} & \ldots & x_{mn} \end{array}\right]$$