SE524/EC524 Optimization Theory and Methods

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Lecture 6: Outline

1. How is the cost affected when we move from bfs to bfs ?
2. Optimality condition.
3. Unboundness criterion.
4. An iteration of the simplex method.
5. Convergence.
6. The full simplex tableau.
7. Example.

How is the cost affected ?

Initially we are at bfs **x** with cost

$$C^{old} = \sum_{i=1}^{m} c_{B(i)} x_{B(i)}$$

Move to new bfs **y** with cost

$$\begin{aligned} C^{new} &= C^{old} + \theta^* c_j - \theta^* \sum_{i=1}^{m} c_{B(i)} x_{ij} \\ &= C^{old} + \theta^* (c_j - c_B' \mathbf{B}^{-1} \mathbf{A}_j) \end{aligned}$$

Definition

(Reduced cost of variable x_j): $\bar{c}_j \triangleq c_j - c_B' \mathbf{B}^{-1} \mathbf{A}_j$

Profitable to move only if $\bar{c}_j < 0$.

Optimality Condition

Theorem

(Optimality Thm.) Let $\bar{\mathbf{c}} = (\bar{c}_1, \ldots, \bar{c}_n)'$, and **x** the current bfs.

1. If $\bar{\mathbf{c}} \geq \mathbf{0} \quad \Rightarrow$ **x** is optimal.
2. If **x** is optimal and nondegenerate $\Rightarrow \bar{\mathbf{c}} \geq \mathbf{0}$.

An iteration of the simplex method

1. **Start** with basic columns $\mathbf{A}_{B(1)}, \dots, \mathbf{A}_{B(m)}$ and associated bfs $\mathbf{x}$.
2. Compute the reduced costs $\bar{c}_j$ for all nonbasic indices. If they are nonnegative, $\mathbf{x}$ is optimal and algorithm terminates. Else pick a j with $\bar{c}_j < 0$.
3. **Bring x_j into basis**: Compute $\mathbf{u} = (x_{1j}, \dots, x_{mj})' = \mathbf{B}^{-1}\mathbf{A}_j$. If no component positive, then $\theta^* = \infty$, optimal cost $= -\infty$, and the algorithm terminates.
4. If some component positive, let $\theta^* = \min_{\{i|x_{ij}>0\}} \frac{x_{B(i)}}{x_{ij}}$
5. Let l the minimizer. Form a new basis by replacing $\mathbf{A}_{B(l)}$ with $\mathbf{A}_j$. If $\mathbf{y}$ the new bfs, then basic variables have values:

$$y_j = \theta^*$$
$$y_{B(i)} = x_{B(i)} - \theta^* x_{ij}, \quad i \neq l$$

Convergence

Theorem

Assume that the feasible set is nonempty and all bfs are nondegenerate. Then the simplex method terminates in a finite # of iterations and either:

(a) We arrive at an optimal bfs, or
(b) We have found a vector $\mathbf{d}$, with $\mathbf{Ad} = \mathbf{0}$, $\mathbf{d} \geq \mathbf{0}$, $\mathbf{c}'\mathbf{d} < 0$, and the optimal cost $= -\infty$.

The full simplex tableau

$-\mathbf{c}_B'\mathbf{x}_B$	$\bar{c}_1$	$\ldots$	$\bar{c}_n$
$x_{B(1)}$	$\vert$		$\vert$
$\vdots$	$\mathbf{B}^{-1}\mathbf{A}_1$	$\ldots$	$\mathbf{B}^{-1}\mathbf{A}_n$
$x_{B(m)}$	$\vert$		$\vert$

=

$-\mathbf{c}_B'\mathbf{x}_B$	$\mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A}$
$\mathbf{B}^{-1}\mathbf{b}$	$\mathbf{B}^{-1}\mathbf{A}$

Example

$$
\begin{array}{llll}
\min & -10x_1 & -12x_2 - 12x_3 & \\
\text{s.t.} & x_1 & +2x_2 + 2x_3 & \leq 20 \\
 & 2x_1 & +x_2 + 2x_3 & \leq 20 \\
 & 2x_1 & +2x_2 + x_3 & \leq 20
\end{array}
$$