SE524/EC524 Optimization Theory and Methods

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Lecture 8: Outline

1. General form of the dual.
2. Weak duality.
3. Strong duality.
4. Geometric interpretation of strong duality.
5. Relations between primal and dual.

Constructing the Dual

Consider the LP with optimal solution $\mathbf{x}^*$

$$\textbf{Primal} \quad \begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Relax the constraint by introducing the vector of **Lagrange multipliers** (or **dual variables**) $\mathbf{p}$

$$g(\mathbf{p}) = \begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Note: $g(\mathbf{p}) \leq \mathbf{c}'\mathbf{x}^*$. Get the tightest lower bound

$$\textbf{Dual} \quad \max g(\mathbf{p}) \quad \Leftrightarrow \quad \begin{aligned} & \max \mathbf{p}'\mathbf{b} \\ & \text{s.t. } \mathbf{p}'\mathbf{A} \leq \mathbf{c}' \end{aligned}$$

General form of the dual

$$\textbf{Primal} \quad \begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} & \\ & \mathbf{a}_i'\mathbf{x} \geq b_i & i \in M_1 \\ & \mathbf{a}_i'\mathbf{x} \leq b_i & i \in M_2 \\ & \mathbf{a}_i'\mathbf{x} = b_i & i \in M_3 \\ & x_j \geq 0 & j \in N_1 \\ & x_j \leq 0 & j \in N_2 \\ & x_j \lessgtr 0 & j \in N_3 \end{aligned} \qquad \textbf{Dual} \quad \begin{aligned} \max \quad & \mathbf{p}'\mathbf{b} & \\ & p_i \geq 0 & i \in M_1 \\ & p_i \leq 0 & i \in M_2 \\ & p_i \lessgtr 0 & i \in M_3 \\ & \mathbf{p}'\mathbf{A}_j \leq c_j & j \in N_1 \\ & \mathbf{p}'\mathbf{A}_j \geq c_j & j \in N_2 \\ & \mathbf{p}'\mathbf{A}_j = c_j & j \in N_3 \end{aligned}$$

Primal	min	max	dual
constraints	$\geq b_i$	≥ 0	variables
	$\leq b_i$	≤ 0	
	$= b_i$	$\lessgtr 0$	
variables	≥ 0	$\leq c_j$	constraints
	≤ 0	$\geq c_j$	
	$\lessgtr 0$	$= c_j$	

Properties

Theorem

The dual of the dual is the primal.

Theorem

(Weak Duality) If $\mathbf{x}$ is primal feasible and $\mathbf{p}$ is dual feasible then $\mathbf{p}'\mathbf{b} \leq \mathbf{c}'\mathbf{x}$.

Corollary

If $\mathbf{x}$ is primal feasible, $\mathbf{p}$ is dual feasible, and $\mathbf{p}'\mathbf{b} = \mathbf{c}'\mathbf{x}$, then $\mathbf{x}$ is optimal in the primal and $\mathbf{p}$ is optimal in the dual.

Theorem

(Strong Duality) If LP has optimal solution, then so does the dual, and the optimal costs are equal.

Relations between primal and dual

	Finite opt.	**Unbounded**	**Infeasible**
Finite opt.	$\surd$	x	x
Unbounded	x	x	$\surd$
Infeasible	x	$\surd$	$\surd$