SE524/EC524 Optimization Theory and Methods

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Lecture 9: Outline

1. Complementary slackness.
2. Geometry of duality.
3. Economic interpretation: shadow prices.
4. Dual simplex.
5. Geometry of dual simplex.

Complementary slackness

Theorem

[Complementary slackness (CS)] Let $\mathbf{x}$ primal feasible and $\mathbf{p}$ dual feasible. Then $\mathbf{x}, \mathbf{p}$ optimal **iff**

$$p_i(\mathbf{a}_i'\mathbf{x} - b_i) = 0, \qquad \forall i$$

$$x_j(c_j - \mathbf{p}'\mathbf{A}_j) = 0, \qquad \forall j$$

Dual simplex

In simplex method $\mathbf{p}' = \mathbf{c}_B'\mathbf{B}^{-1}$ (by proof of strong duality). **Primal optimality condition** is

$$\mathbf{c}' - \mathbf{c}_B'\mathbf{B}^{-1}\mathbf{A} \geq \mathbf{0}$$

which is the same as **dual feasibility**.

This is a **primal algorithm**: maintains **primal feasibility** and works towards **dual feasibility**.

Dual algorithm: maintains **dual feasibility** and works towards **primal feasibility**.

Dual simplex: Move from dual bfs to dual bfs.

Dual simplex (cont.)

$-\mathbf{c}'_B\mathbf{x}_B$	$\bar{c}_1$	$\ldots$	$\bar{c}_n$
$x_{B(1)}$	$\vert$		$\vert$
$\vdots$	$\mathbf{B}^{-1}\mathbf{A}_1$	$\ldots$	$\mathbf{B}^{-1}\mathbf{A}_n$
$x_{B(m)}$	$\vert$		$\vert$

Do not require $\mathbf{B}^{-1}\mathbf{b} \geq 0$, i.e., we are at a basic solution of the primal (not bfs).
Assume $\bar{\mathbf{c}} \geq \mathbf{0}$ (dual feasibility).
Dual cost is

$$\mathbf{p}'\mathbf{b} = \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}'_B\mathbf{x}_B$$

i.e., the negative of upper left corner.
If $\mathbf{B}^{-1}\mathbf{b} \geq 0$ then both dual feasibility and primal feasibility, and also same cost $\Rightarrow$ **optimality**.
Otherwise, change basis.

An iteration of dual simplex

1. Start with basis matrix $\mathbf{B}$ and all reduced costs ≥ 0.
2. If $\mathbf{B}^{-1}\mathbf{b} \geq 0$ have an optimal solution, else choose l s.t. $x_{B(l)} < 0$.
3. Consider the lth row (pivot row) $x_{B(l)}, v_1, \ldots, v_n$. If $\forall i\ v_i \geq 0$ then dual optimal cost $= +\infty$ and algorithm terminates.
4. Else, let j s.t.
$$\frac{\bar{c}_j}{|v_j|} = \min_{\{i \mid v_i < 0\}} \frac{\bar{c}_i}{|v_i|}$$
Pivot element v_j. $\mathbf{A}_j$ enters the basis and $\mathbf{A}_{B(l)}$ exits.

Anticycling

If $\bar{c}_j = 0$ then we don't move and algorithm can cycle. To avoid this

Lexicographic pivoting
To find pivot element, for each column with $v_i < 0$ divide all its entries by $|v_i|$ and choose lexicographically smallest column. If there is a tie choose column with smallest index.

When should we use the dual simplex ? When is more convenient. Suppose that we used primal simplex to solve a problem, and we found the optimal. Say we now want to re-solve the problem with a new $\mathbf{b}$. Reduced costs of the original problem provide dual bfs for the new problem.