Lecture 10: Outline

- Lagrange multipliers: necessary optimality conditions for problems with equality constraints.
Problems with equality constraints

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad h_i(x) = 0, \quad i = 1, \ldots, m
\end{align*}
\]

where \( f, h_i \) are continuously differentiable (in a open set containing the minimum).

Letting \( \mathbf{h} = (h_1, \ldots, h_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) we have

\[
\begin{align*}
\min & \quad f(x) \\
\text{s.t.} & \quad \mathbf{h}(x) = 0
\end{align*}
\]

Necessary optimality conditions

**Proposition**

Let \( \mathbf{x}^* \) be a local minimum. Assume that \( \mathbf{x}^* \) is **regular**, i.e., \( \nabla h_1(\mathbf{x}^*), \ldots, \nabla h_m(\mathbf{x}^*) \) are linearly independent. Then, there exists a Lagrange multiplier vector \( \mathbf{\lambda}^* = (\lambda_1^*, \ldots, \lambda_m^*) \) s.t.

\[
\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla h_i(\mathbf{x}^*) = 0.
\]

If \( f, \mathbf{h} \) are twice continuously differentiable

\[
\begin{align*}
\mathbf{y}' \left( \nabla^2 f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla^2 h_i(\mathbf{x}^*) \right) & \geq 0, \quad \forall \mathbf{y} \in V(\mathbf{x}^*),
\end{align*}
\]

where \( V(\mathbf{x}^*) = \{ \mathbf{y} | \nabla h_i(\mathbf{x}^*)' \mathbf{y} = 0, \ i = 1, \ldots, m \} \) is the subspace of first order feasible directions.