Lecture 2: Outline

1. Background material: subspaces, polyhedra, hulls, norms, continuity, limits.
2. Convexity.
Polyhedral Geometry: Some Definitions

**Polyhedron** is a set of the form \( \{ x \in \mathbb{R}^n \mid Ax \geq b \} \).

**Polyhedron in standard form** is a set of the form \( \{ x \in \mathbb{R}^n \mid Ax = b, \ x \geq 0 \} \).

**Polytope** is a bounded polyhedron (i.e., \( \exists K \) s.t. \( |x_i| \leq K \) \( \forall x = (x_1, \ldots, x_n) \in \text{polyhedron} \)).

**Hyperplane** is a set of the form \( \{ x \in \mathbb{R}^n \mid a'x = b \} \).

**Halfspace** is a set of the form \( \{ x \in \mathbb{R}^n \mid a'x \geq b \} \).

A polyhedron is the intersection of a finite number of halfspaces \( (Ax \geq b \Rightarrow a'_ix \geq b_i, \ i = 1, \ldots, m) \).

\( S \subset \mathbb{R}^n \) is **convex** if for any \( x, y \in S \) and any \( \lambda \in [0, 1] \):
\[
\lambda x + (1 - \lambda)y \in S.
\]
Let \( \lambda_i \geq 0 \sum_{i=1}^{k} \lambda_i = 1 \). The vector \( \sum_{i=1}^{k} \lambda_i x_i \) is said to be **convex combination** of \( x_i \)’s.

**Convex hull** of \( x_i \)'s is the set of all their convex combinations.

Properties of Convex Sets

**Theorem**

1. The intersection of convex sets is convex.
2. Polyhedra are convex sets.
3. Convex combination of finite number of elements of a convex set belongs to the set.
4. Convex hull of a finite number of vectors is a convex set.
5. The vector sum \( \{ x_1 + x_2 \mid x_1 \in \mathcal{C}_1, x_2 \in \mathcal{C}_2 \} \) of convex sets \( \mathcal{C}_1, \mathcal{C}_2 \) is convex.
6. The image of a convex set under a linear transformation is convex.
Real Analysis background

- Norms $\| \cdot \|$ on $\mathbb{R}^n$.
- Euclidean norm: $\| x \| = \sqrt{x'x}$.
- Limits.
- Open Ball around $a$ with radius $r$: $\{ y \mid \| y - a \| < r \}$.
- $A \subset \mathbb{R}^n$ is compact iff closed and bounded (Heine-Borel).
- Consider function $f : A \rightarrow \mathbb{R}^n$:
  - **continuous** at $x \in A$ if $\lim_{y \rightarrow x} f(y) = f(x)$.
  - **right-continuous** if $\lim_{y \downarrow x} f(y) = f(x)$.
  - **left-continuous** if $\lim_{y \uparrow x} f(y) = f(x)$.
  - **lower-semicontinuous** if $f(x) \leq \liminf_{k \rightarrow \infty} f(x_k)$ for every sequence $x_k \rightarrow x$.
  - **upper-semicontinuous** if $f(x) \geq \limsup_{k \rightarrow \infty} f(x_k)$ for every sequence $x_k \rightarrow x$.
  - **coercive** if $\lim_{k \rightarrow \infty} f(x_k) = \infty$ for every sequence satisfying $\| x_k \| \rightarrow \infty$. 