Lecture 3: Outline

- **Corner Points:**
  - Extreme point.
  - Vertex.
  - Basic Feasible Solution (BFS).

- **Theorem:** Extreme point ⇔ Vertex ⇔ BFS.

- BFS in standard form polyhedra.
- Degeneracy is standard form polyhedra.
Extreme Points

Definition

Let $P$ a polyhedron. A vector $x \in P$ is an extreme point if $\nexists \ y, z \in P$ and $y \neq x$ s.t. $x$ can be written as a convex combination of $y, z$.

Vertex

Definition

A vector $x \in P$ is a vertex if $\exists \ c$ s.t. $x$ is the unique minimizer of $c'y$ over $P$.  

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Basic Feasible Solution (BFS)

We call active or binding the constraints that are satisfied with equality.

**Definition**

Let \( x^* \in \mathbb{R}^n \), and a polyhedron \( P \).

1. \( x^* \) is a basic solution if
   - All equality constraints are active.
   - Let \( I = \{ i \mid a^i x^* = b_i \} \). The subspace spanned by \( \{ a_i \mid i \in I \} \) is all of \( \mathbb{R}^n \).

2. If \( x^* \) is feasible then is called a basic feasible solution (bfs).

A basic solution \( x^* \) is called degenerate if \( |I| > n \), and nondegenerate otherwise.

Equivalence of corner point definitions

**Theorem**

Extreme point ⇔ Vertex ⇔ BFS.

**Proof:**

1. Vertex ⇒ Extreme point.
2. Extreme point ⇒ BFS.
3. BFS ⇒ Vertex.

Two basic solutions are adjacent if there are \( n - 1 \) linearly independent constraints active at both of them.

If two bfs are adjacent, the line segment that connects them is called an edge.
BFS in standard form polyhedra

Let $P = \{ x \mid Ax = b, x \geq 0 \}$ nonempty. WLOG assume $A$ has $m$ linearly independent rows.

**Example**

\[
\begin{align*}
x_1 + 2x_2 &= 2 \\
2x_1 + x_2 &= 4 \\
2x_1 + 4x_2 &= 4
\end{align*}
\]

For a BFS we need $n$ active “lin. ind.” constraints: $Ax = b$ provides $m$. Hence, $n - m$ of $x \geq 0$ should be active.

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BFS in standard form polyhedra: Construction

Pick $m$ linearly independent columns of $A$ (wlog 1, 2, \ldots, $m$).

\[
m\{ \begin{bmatrix} B \mid N \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b \} \quad \Rightarrow \quad \begin{cases} x_B = B^{-1}b \\ x_N = 0 \end{cases}
\]

- $x^* = (x_B, x_N)$ is a **basic solution**.
- If $x^* \geq 0$ it is a **basic feasible solution**.
- The elements of $x_B$ are called **basic variables**.
- The elements of $x_N$ are called **nonbasic variables**.
- $B$ is called a **basis matrix**, its columns form a **basis** (of $\mathbb{R}^m$).
- Two bases that share $m - 1$ columns are called **adjacent** and they correspond to adjacent bfs.
- A basis **uniquely** determines a bfs, **but** different bases may lead to the same bfs.
Degeneracy is standard form polyhedra

**Definition**

When the set of active constraints at a BFS has more than $n$ elements then the BFS is **degenerate**.

In **standard form**: $Ax = b$ are $m$ active constraints. If more that $n - m$ of $x \geq 0$ active then **degenerate bfs**.

The following statements are equivalent and imply **degeneracy**:

- More than $n - m$ variables $= 0$.
- Some basic variables $= 0$. 